DRAWDOWN MEASURE IN PORTFOLIO OPTIMIZATION

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Abstract

We propose a new one-parameter family of risk measures called Conditional Drawdown (CDD). These measures of risk are functionals of the portfolio drawdown (underwater) curve considered in an active portfolio management. For some value of the tolerance parameter α, in the case of a single sample path, drawdown functional is defined as the mean of the worst $(1 - \alpha) * 100\%$ drawdowns. The CDD measure generalizes the notion of the drawdown functional to a multi-scenario case. The CDD measure includes the Maximal Drawdown and Average Drawdown as its limiting cases. We studied mathematical properties of the CDD and developed efficient optimization techniques for CDD computation and solving asset allocation problems with CDD measure. For a particular example, we find the optimal portfolios for a case of Maximal Drawdown, a case of Average Drawdown, and several intermediate cases between these two. The CDD family of risk functionals is similar to Conditional Value-at-Risk (CVaR), which is also called Mean Shortfall, Mean Access loss, or Tail Value-at-Risk. Some recommendations on how to select the optimal risk functionals for getting practically stable portfolios are provided. We solved a real life portfolio allocation problem using the proposed measures.

1 Introduction

Optimal portfolio allocation is a longstanding issue in both practical portfolio management and academic research on portfolio theory. Various methods have been proposed and studied (for a recent review, see, for example, [11]). All of them, as a starting point, assume some measure of portfolio performance, which consists of at least two components: *a)* evaluating expected portfolio reward; and *b)* assessing expected portfolio risk. From theoretical prospective, there are two well-known approaches to manage portfolio performance: *Expected Utility Theory* and *Risk Management*, which are usually considered within a framework of a one-period or multi-period model.

If we are interested in Risk Management approach to portfolio optimization within a long term, what are the functionals for assessing portfolio risk, which account for different sequences of portfolio losses? Let portfolio be optimized within time interval $[0, T]$, and let $W(t)$ be portfolio value at time

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moment $t \in [0, T]$. One of the functionals that we are looking for is portfolio *drawdown* defined by $\left(\max_{\tau\in[0,t]} W(\tau) - W(t)\right) \Big/ W(t)$, which, indeed, accounts for a sequence of portfolio losses. What are the advantages to formulate a portfolio optimization problem with a constraint on portfolio drawdown? In order to answer to all these questions, drawdown regulations in real trading strategies and drawdown theoretical aspects are addressed.

Drawdown regulations in real trading strategies. From a standpoint of a fund manager, who trades clients' or bank's proprietary capital, and for whom the clients' accounts are the only source of income coming in the form of management and incentive fees, losing these accounts is equivalent to the death of his business. This is true with no regard to whether the employed strategy is long-term valid and has very attractive expected return characteristics. Such fund manager's primary concern is to keep the existing accounts and to attract the new ones in order to increase his revenues. Commodity Trading Advisor (CTA) determines the following rules regarding magnitude and duration of their clients' accounts drawdowns:

- *highly unlikely to tolerate a 50% drawdown in an account with an average- or small-risk CTA;*
- *an account may be shut down if a 20% drawdown is breached;*
- *a warning is issued if an account in a 15% drawdown;*
- *an account will be closed if it is in a drawdown, even of small magnitude, for longer than 2 years;*
- *time to get out of a drawdown should not be longer than a year.*

Drawdown notion in theoretical framework. Several papers discussed portfolio optimization with drawdown constraints. First, we want to mention the paper [12], where an exact analytical solution to portfolio optimization with constraint on maximal drawdown is obtained based on the following model:

- *continuous setup;*
- *one-dimensional case — allocating current capital among one risky and one risk-free assets;*
- *an assumption of log-normality of the risky asset;*
- *use of dynamic programming approach — finding a time-dependent fraction of the current capital invested into the risky asset.*

Second, a subsequent generalization of [12] for multi-dimension case (several risky assets) was done in [7].

In contrast to [7, 12], the paper [6] defined portfolio drawdown to be the drop of the current portfolio value comparing to its maximum achieved in the past up to current moment t, i.e. $\max_{\tau\in[0,t]} W(\tau) - W(t)$, and introduced one-parameter family of drawdown functionals, entitled Conditional Drawdown (CDD). Moreover, work [6] considered portfolio optimization with a constraint on the drawdown functionals in a setup similar to the index tracking problem discussed in [8], where an index historical performance is replicated by a portfolio with constant weights, namely

- *discrete formulation;*
- *multi-dimensional case — several risky assets (markets and futures);*
- *a constant set of portfolio weights satisfying a certain risk condition over the whole interval* $[0, T]$*;*
- *no assumption about the underlying probability distribution, which allows considering variety of practical applications — use of the historical sample paths of assets' rates of returns over* $[0, T]$*;*
- *use of linear programming approach — reduction of portfolio optimization to linear programming (LP) problem.*

The CDD is related to Value-at-Risk (VaR) risk measure and to Conditional Value-at-Risk (CVaR) risk measure studied in [18, 19]. By definition, with respect to a specified probability level α, the α-VaR of a portfolio is the lowest amount ζ_α such that, with probability α, the loss will not exceed ζ_α in a specified time τ (see, for instance, [10]), whereas the α-CVaR is the conditional expectation of losses above that amount ζ_α. The CDD is similar to CVaR and can be viewed as a modification of the CVaR to the case when the loss-function is defined as a drawdown. CDD and CVaR are conceptually closely related percentile-based risk performance functionals. Optimization approaches developed for CVaR are directly extended to CDD. The CDD contains the average drawdown and maximal drawdown as its limiting cases. It takes into account both the magnitude and duration of the drawdowns, whereas the maximal drawdown concentrates on a single event — maximal account's loss from its previous peak.

However, the paper [6] was only the test for suggested approach to portfolio optimization subject to constraints on drawdown functionals. Conditional Drawdown introduced in [6] was not defined as a true risk measure and the real-life portfolio optimization example was considered based only on the historical sample paths of assets' rates of returns.

This paper presents:

- *concept of drawdown measure — possession of all properties of a deviation measure, generalization of deviation measures to a dynamic case;*
- *concept of risk profiling — Mixed Conditional Drawdown (direct generalization of CDD);*
- *optimization techniques for CDD computation — reduction to linear programming (LP) problem;*
- *portfolio optimization with constraint on Mixed CDD.*

This paper develops concept of drawdown measure by generalizing the notion of Conditional Drawdown to the case of several sample paths for portfolio uncompounded rate of return. Definition of drawdown measure is essentially based on the notion of Conditional Value-at-Risk (CVaR) [1, 18, 19] and *mixed* CVaR [20] extended to a multi-scenario case. Drawdown measure uses the same concept of risk profiling introduced in [20], namely, drawdown measure is a "multi-scenario" *mixed* CVaR applied to drawdown loss-function.

From theoretical prospective, CDD measure satisfies the system of axioms determining *deviation measures* [20, 21]. Those axioms are: *nonnegativity, insensitivity to constant shift, positive homogeneity* and *convexity*. Moreover, CDD measure is an example generalizing properties of deviation measures to a dynamic case. We develop optimization techniques for efficient computation of CDD measure in the case when instruments' rates of returns are given.

Similar to the Markowitz mean-variance approach [14], we formulate and solve an optimization problem with the reward performance function and CDD constraints. The reward-CDD optimization problem is a piece-wise linear convex optimization problem (see definition of convexity in [17]), which can be reduced to a linear programming problem using auxiliary variables.

Linear programming allows solving large optimization problems with hundreds of thousands of instruments. The algorithm is fast, numerically stable, and provides a solution during one run (without adjusting parameters like in genetic algorithms or neural networks). Linear programming approaches are routinely used in portfolio optimization with various criteria, such as mean absolute deviation [13], maximum deviation [23], and mean regret [8]. A reader interested in other applications of optimization techniques in the finance area can find relevant papers in [24].

2 Model development

Suppose a given time interval $[0, T]$ is partitioned into N subintervals $[t_{k-1}, t_k]$, $k = \overline{1, N}$, by the set of points $\{t_0 = 0, t_1, t_2, \ldots, t_N = T\}$, and suppose there are m risky assets with rates of returns determined by *random* vector $r(t_k) = (r_1(t_k), r_2(t_k), \ldots, r_m(t_k))$ at time moments t_k for $k = \overline{1, N}$. We also assume that the risk-free instrument (or cash) with the constant rate of return r_0 is available. The i^{th} asset's rate of return at time moment t_k is defined by $r_i(t_k) = \frac{p_i(t_k)}{p_i(t_{k-1})} - 1$, where $p_i(t_k)$ and $p_i(t_{k-1})$ are the i^{th} asset's prices per share at moments t_k and t_{k-1}, respectively. Let C denote an initial capital at $t_0 = 0$ and let values $x_i(t_k)$ for $i = \overline{1, m}$ and $x_0(t_k)$ define the proportion of the current capital invested in the i^{th} risky asset and risk-free instrument at t_k, respectively. Consequently, a portfolio formed of the m risky assets and the risk-free instrument is determined by the vector of weights $x(t_k) = (x_0(t_k), x_1(t_k), x_2(t_k), \ldots, x_m(t_k))$. The components of $x(t_k)$ satisfy the budget constraint

$$\sum_{i=0}^{m} x_i(t_k) = 1. \tag{1}$$

By definition, the rate of return of the portfolio at time moment t_k is

$$r_k^{(p)}(x(t_k)) = r(t_k) \cdot x(t_k) = \sum_{i=0}^{m} r_i(t_k)\, x_i(t_k). \tag{2}$$

Portfolio optimization can be considered within a framework of a one-period or multi-period model. A *one-period* model in portfolio optimization assumes the i^{th} asset's rates of returns for all t_k, $k = \overline{1, N}$, to be independent observations of a random variable r_i. In this case, the vector of portfolio weights is constant and the portfolio's rate of return is a random variable $r^{(p)}$ presented by a linear combination of random assets' rates of returns r_i, $i = \overline{1, m}$, and constant r_0, i.e. $r^{(p)} = \sum\limits_{i=0}^{m} r_i\, x_i$. A traditional setup for a one-period portfolio optimization problem from Risk Management point of view is maximizing portfolio expected rate of return subject to the budget constraint and a constraint on risk, i.e.

$$\begin{array}{ll} \max\limits_{x} & E(r^{(p)}) \\ s.\,t. & \mathcal{R}isk(r^{(p)}) \le d, \\ & \sum\limits_{i=0}^{m} x_i = 1. \end{array} \tag{3}$$

Risk of the portfolio can be measured by different performance functions, depending on investor's risk preferences. Variance, Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Mean Absolute Deviation (MAD) are examples of risk measures used in Portfolio Risk Management (see ...). Certainly, solving optimization problem (3) with different risk measures will lead to different optimal portfolios. However, all of them are based on a one-period model, which does not take into account the sequence of the asset's rates of returns within the time interval $[0, T]$.

A *multi-period* model in portfolio optimization is intended for controlling and optimizing portfolio wealth over a long term. It is essentially based on how the asset's rates of returns evolve within a whole time interval. Moreover, in each time moment t_k, $k = \overline{0, N}$, there might be a capital inflow or outflow into or from the portfolio and portfolio weights $x_i(t_k)$, $i = \overline{1, m}$, might be rebalanced. In this case, the portfolio wealth at t_k for $k = \overline{1, N}$ is defined

$$W_k(x(t_k)) = (W_{k-1}(x(t_{k-1})) + Y(t_{k-1}))\,(1 + r_k^{(p)}(x(t_k))), \tag{4}$$

where $Y(t_{k-1}) = F_+(t_{k-1}) - F_-(t_{k-1})$ is the resulting capital flow at t_{k-1} (inflow $F_+(t_{k-1})$ minus outflow $F_-(t_{k-1})$), which can be positive or negative.

A portfolio optimization problem can be formulated based on the Expected Utility Theory (EUT) or Risk Management approach. According to the EUT, an investor with additively separable concave utility function $U(.)$ chooses a consumption stream $\{C_0, C_1, \ldots, C_{N-1}\}$ and portfolio to maximize $E\left(\sum_{k=0}^{N-1} U\left(C(t_k), t_k\right) + B\left(W(t_N, x(t_N))\right)\right)$, where $B(.)$ is the concave utility of bequest. Note the EUT is focused on maximization of investor's consumption. However, a risk manager who runs a hedge fund and wishes to increase capital inflow by attracting new investors would be more interested in maximizing portfolio wealth at the final moment $t_N = T$ and decreasing portfolio drops over the whole time interval $[0, T]$. In this case, Risk Management approach is quite adequate to formulate a portfolio optimization problem

$$\begin{aligned} &\max_x \ \mathcal{P}(W) \\ &s.\,t. \quad \mathcal{R}(W) \le d, \\ &\qquad \sum_{i=0}^{m} x_i(t_k) = 1, \quad k = \overline{0, N}, \end{aligned} \tag{5}$$

where $\mathcal{P}(W)$ and $\mathcal{R}(W)$ are performance and risk functionals, respectively, depending on stream $W = (W_1,\, W_2,\, \ldots,\, W_N)$.

Suppose the optimization problem (5) is considered under the following conditions:

- *the manager does not affect a stream of* $Y(t_k)$ *(if the portfolio value increases it is likely that capital inflow will also increase and vise-versa);*
- *the manager can only allocate resources among different instruments (investment strategies) in the portfolio at every moment* t_k, $k = \overline{0, N}$, *i.e. he/she can only optimize portfolio rate of return by choosing portfolio weights* $x_i(t_k)$.

Accounting for these conditions, how can the manager evaluate portfolio performance over $[0, T]$ and efficiently solve (5)? Before to answer to this question, the following legitimate issues regarding problem formulation (5) should be addressed:

- *how the risk is measured within* $[0, T]$;
- *how the assets' rates of returns are modeled within* $[0, T]$;
- *what optimization approach is chosen to solve (5).*

These issues are discussed below.

Dynamic performance functionals. Does using variance, VaR, CVaR or MAD by itself make sense in a dynamic case? The obvious answer is no, since no one of them by itself takes into account the sequence of assets' rates of returns. Although, the aforementioned risk measures may be quite appropriate if they are applied to a random variable or functional, which distinguishes different sequences of W_k in a stream $(W_1,\, W_2,\, \ldots,\, W_N)$. One of the functionals accounting for a sequence of W_k is based on the notion of portfolio drawdown, which deals with the drop in portfolio wealth at time moment t_k with respect to the wealth's maximum value preceding t_k. By definition, the portfolio drawdown at time moment t_k is the ratio of the drop in portfolio wealth at t_k to the preceding wealth's maximum

$$\mathcal{DD}\left(W, t_k\right) = 1 - \frac{W_k(x(t_k))}{\max\limits_{0 \le j \le k} \{W_j(x(t_j))\}}. \tag{6}$$

One of the possible problem formulations is to consider portfolio optimization with drawdown constraints in continuous dynamics [7], [12].

$$\Pr\left\{\mathcal{DD}\left(W, t\right) \le \gamma, \ \ \forall\, t \in [0, T]\right\} = 1, \tag{7}$$

that is, portfolio drawdown should not exceed given value $\gamma \in [0,1]$ almost surely for all $t \in [0,T]$. However, instead of imposing the constraint (7) for all $t \in [0,T]$, we are interested in maximizing portfolio expected rate of return while controlling an integral characteristic of portfolio performance. We entitle such a characteristic to be a dynamic performance functional (DPF). In this case, following Risk Management methodology, we would be able to construct an efficient frontier establishing the dependence between the expected rate of return of optimal portfolios and corresponding values of the DPF.

Developing dynamic performance functionals based on the notion of portfolio drawdown and solving a real-life portfolio optimization problem with these functionals is the subject of this paper. The notion of absolute drawdown and development of the DPF based on this notion are presented in the next section.

3 The Absolute Drawdown and Dynamic Performance Functionals in the case of a single sample path

This section presents the notion of the Absolute Drawdown ($\mathcal{AD}$) and considers three Dynamic Performance Functions (DPF) based on this notion. The $\mathcal{AD}$ is applied to a sample path of the uncompounded cumulative portfolio rate of return. Note the $\mathcal{AD}$ is applied not to the compounded cumulative portfolio rate of return $W_k(x(t_k))$. If the values of $r_k^{(p)}(x(t_k))$ for $k = \overline{1,N}$ determine a sample path (time series) of the portfolio's rate of return, then, by definition, the *uncompounded cumulative* portfolio rate of return at time moment t_k is

$$w_k(x(t_k)) = \begin{cases} 0, & k = 0, \\ \sum_{l=1}^{k} r_l^{(p)}(x(t_l)), & k = \overline{1,N}. \end{cases} \tag{8}$$

To simplify notations, we use w_k instead of $w_k(x(t_k))$, assuming that w_k is always a function of vector $x(t_k)$. Further in this section, we consider only a single sample path of w_k, $k = \overline{1,N}$, which we denote by vector w, i.e. $w = (w_1, \ldots, w_N)$.

Definition 1. *The $\mathcal{AD}$ is a vectorial-functional depending on the sample path w*

$$\mathcal{AD}(w) = \xi = (\xi_1, \ldots, \xi_N), \quad \xi_k = \max_{0 \le j \le k}\{w_j\} - w_k. \tag{9}$$

Note that components $(w_1, \ldots, w_N)$ and $(\xi_1, \ldots, \xi_N)$ of vectors w and ξ, are, in fact, time series $w_1, \ldots, w_N$ and $\xi_1, \ldots, \xi_N$, respectively, where the k^{th} components of w and ξ correspond to time moment t_k. Since ξ_0 is always zero, we do not include it into drawdown time series ξ. Moreover, although $\mathcal{AD}(w)$ and ξ are the same drawdown time series, we refer to notation $\mathcal{AD}(w)$ to emphasize its dependence on w and to notation ξ whenever we use drawdown time series just as vector of numbers.

Figure 1 illustrates an example of the absolute drawdown ξ and a corresponding sample path of uncompounded cumulative rate of return w. Starting from $t_0 = 0$, uncompounded cumulative rate of return w goes up and the first component of ξ equals zero. When w decreases, ξ goes up. When time series w achieves its local minimum, absolute drawdown achieves its local maximum. This process continues until $t_N = T$.

Proposition 1. *Defining vectorial operations: $w + const = (w_1 + const, \ldots, w_N + const)$ and $\lambda w = (\lambda w_1, \ldots, \lambda w_N)$, the $\mathcal{AD}(w)$ satisfies the following properties*

1. Nonnegativity: $\mathcal{AD}(w) \ge 0$.

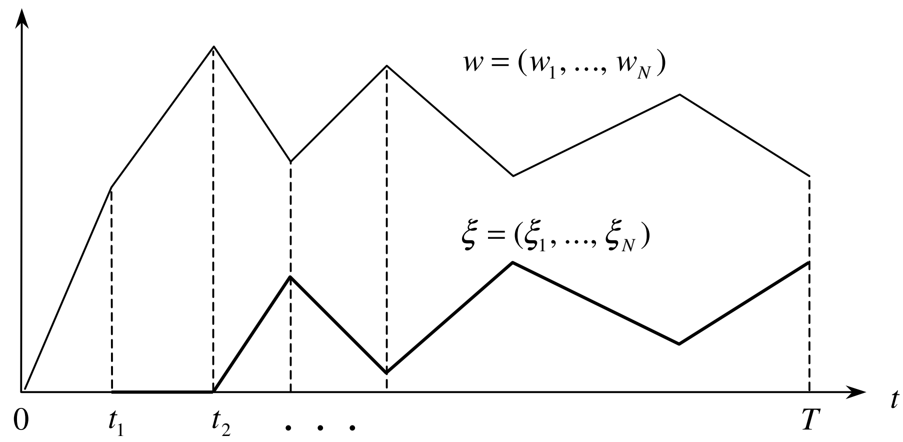

Figure 1: Time series of uncompounded cumulative rate of return w and corresponding absolute drawdown ξ.

2. Insensitivity to constant shift: $\mathcal{AD}(w + const) = \mathcal{AD}(w)$.

3. Positive homogeneity: $\mathcal{AD}(\lambda\, w) = \lambda\, \mathcal{AD}(w)$, $\forall \lambda \geq 0$.

4. Convexity: if $w_\lambda = \lambda\, w_a + (1-\lambda)\, w_b$ *is a linear combination of any two sample paths of uncompounded cumulative rates of returns,* w_a *and* w_b, *with* $\lambda \in [0,1]$, *then* $\mathcal{AD}(w_\lambda) \leq \lambda\, \mathcal{AD}(w_a) + (1-\lambda)\, \mathcal{AD}(w_b)$.

Proof. Properties $1-3$ are direct consequences of definition (9). Property 4 is proved using inequality $\max\limits_{0\leq\tau\leq t} \{\lambda\, w_a + (1-\lambda)\, w_b\} \leq \lambda \max\limits_{0\leq\tau\leq t} \{w_a\} + (1-\lambda) \max\limits_{0\leq\tau\leq t} \{w_b\}$, $\lambda \in [0,1]$. $\square$

Note $\mathcal{DD}$ does not satisfies the properties which $\mathcal{AD}$ does (advantage of $\mathcal{AD}$). The difference between the $\mathcal{AD}$ and $\mathcal{DD}$ is similar to the difference between absolute and relative errors in a measurement. The $\mathcal{AD}$ and $\mathcal{DD}$ functions can be used in Risk Management and Statistics to control absolute and relative drops in a realization of a stochastic process. However, in this paper we are focused on applications of drawdown functionals in Portfolio Optimization.

Issues, which should addressed: 1) reference to [12] and discussion of their example; 2) advantages of using $\mathcal{AD}$ instead of $\mathcal{DD}$ from practical point of view (from theoretical point of view, $\mathcal{DD}$ does not satisfy properties 1-4, which are crucial for further development of risk measures).

Further in this paper we deal only with the absolute drawdown functional and, thus, the word "absolute" is omitted without confusion.

3.1 Maximum, Average and Conditional Drawdowns.

We consider three DPF based on the notion of drawdown: (i) Maximum Drawdown (MaxDD), (ii) Average Drawdown (AvDD), and (iii) Conditional Drawdown (CDD). The last risk functional, Conditional Drawdown, is actually a family of performance functions depending upon a parameter α. It is defined similar to Conditional Value-at-Risk studied in [19] and, as special cases, includes the Maximum Drawdown and the Average Drawdown risk functionals.

Definition 2. *On the time interval* $[0,T]$, *partitioned into* N *subintervals* $[t_{k-1}, t_k]$, $k = \overline{1,N}$, *with* $t_0 = 0$ *and* $t_N = T$, *maximum and average drawdown functionals are defined, respectively*

$$\mathrm{MaxDD}(w) = \max_{1\leq k\leq N} \{\xi_k\}\,, \tag{10}$$

$$\text{AvDD}(w) = \frac{1}{N}\sum_{k=1}^{N} \xi_k. \tag{11}$$

In order to define Conditional Value-@-Risk (CV@R) and Conditional Drawdown (CDD), we introduce a function $\pi_\xi(s)$ such that

$$\pi_\xi(s) = \frac{1}{N}\sum_{k=1}^{N} I_{\{\xi_k \le s\}}, \tag{12}$$

where $I_{\{\xi_k \le s\}}$ is an indicator function equal to 1, if the condition in curly brackets is true, and equal to zero, if the condition is false, i.e.

$$I_{\{c\le s\}} = \begin{cases} 1, & c \le s, \\ 0, & c > s, \end{cases} \qquad c \in \mathbf{R}.$$

Figure (2) explains definition of function $\pi_\xi(s)$. For the threshold s shown on the figure, function $\pi_\xi(s)$ equals $\frac{5}{8}$, since $\xi_k \le s$ for 5 values of k, namely, $k = 2, 3, 4, 7, 8$.

The inverse function to (12) is defined

$$\pi_\xi^{-1}(\alpha) = \begin{cases} \inf \{ \, s \mid \pi_\xi(s) \ge \alpha \} \,, & \alpha \in (0, 1], \\ 0, \quad \alpha = 0. \end{cases} \tag{13}$$

Remark 1. Since all ξ_k, $k = \overline{1, N}$, are nonnegative, we define $\pi_\xi^{-1}(0)$ to be zero.

Remark 2. In fact, $\forall \, \alpha \in (0, 1]$, $s = \pi_\xi^{-1}(\alpha)$ is the unique solution to the two equalities

$$\pi_\xi(s - 0) < \alpha \le \pi_\xi(s + 0). \tag{14}$$

Figures 3 and 4 illustrate left and right continuous step functions $\pi_\xi(s)$ and $\pi_\xi^{-1}(\alpha)$, respectively, which correspond to drawdown time series ξ shown on Figure (2).

Let $\zeta(\alpha)$ denote a threshold such that $(1 - \alpha) * 100\%$ of drawdowns exceed this threshold. By definition,

$$\zeta(\alpha) = \pi_\xi^{-1}(\alpha). \tag{15}$$

If we are able to precisely count $(1 - \alpha) * 100\%$ of the worst drawdowns, then $\pi_\xi(\zeta(\alpha)) = \pi_\xi\left(\pi_\xi^{-1}(\alpha)\right) = \alpha$. For this value of the parameter α, the CV@R of ξ_k, $k = \overline{1, N}$, is defined as the mean of the worst $(1 - \alpha) * 100\%$ drawdowns. For instance, if $\alpha = 0$, then CV@R is the average drawdown, and if $\alpha = 0.95$, then CV@R is the average of the worst 5% drawdowns. However, in a general case, $\pi_\xi(\zeta(\alpha)) = \pi_\xi\left(\pi_\xi^{-1}(\alpha)\right) \ge \alpha$, followed from definition (13). It means that, in general, we are not able to precisely count $(1 - \alpha) * 100\%$ of the worst drawdowns. In this case, the CV@R, is a weighted average of the threshold $\zeta(\alpha)$ and the mean of the worst drawdowns strictly exceeding $\zeta(\alpha)$.

Definition 3. *For a given sequence of ξ_k, $k = \overline{1, N}$, Conditional Value-@-Risk (CV@R) is formally defined by*

$$\text{CV@R}_\alpha(\xi) = \left(\frac{\pi_\xi\left(\zeta(\alpha)\right) - \alpha}{1 - \alpha}\right)\zeta(\alpha) + \frac{1}{(1 - \alpha)N}\sum_{\xi_k \in \, \Xi_\alpha} \xi_k, \tag{16}$$

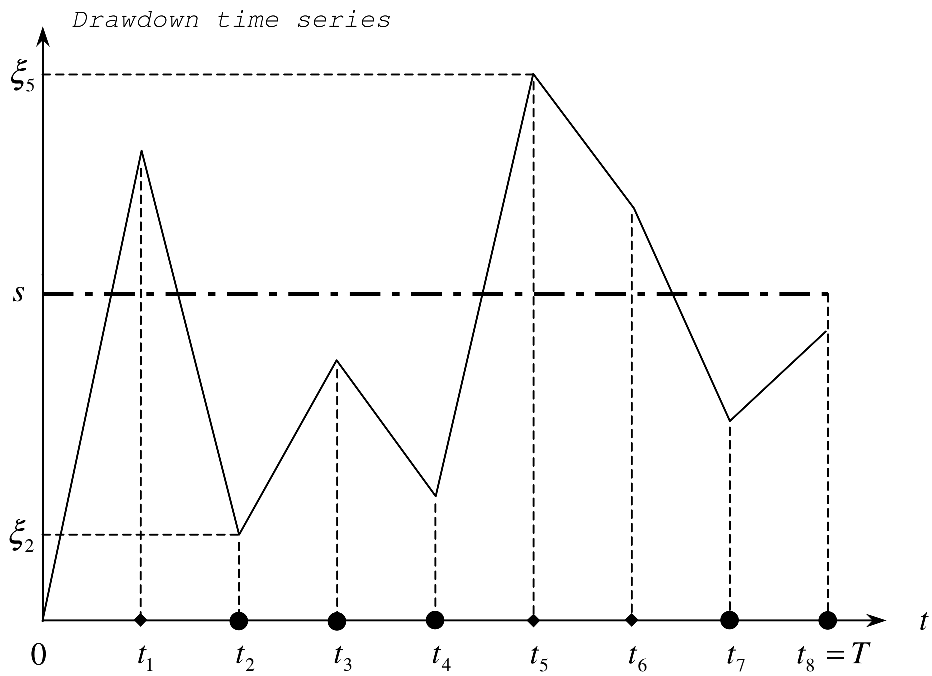

Figure 2: Drawdown time series ξ and indicator function $I_{\{c \le s\}}$.

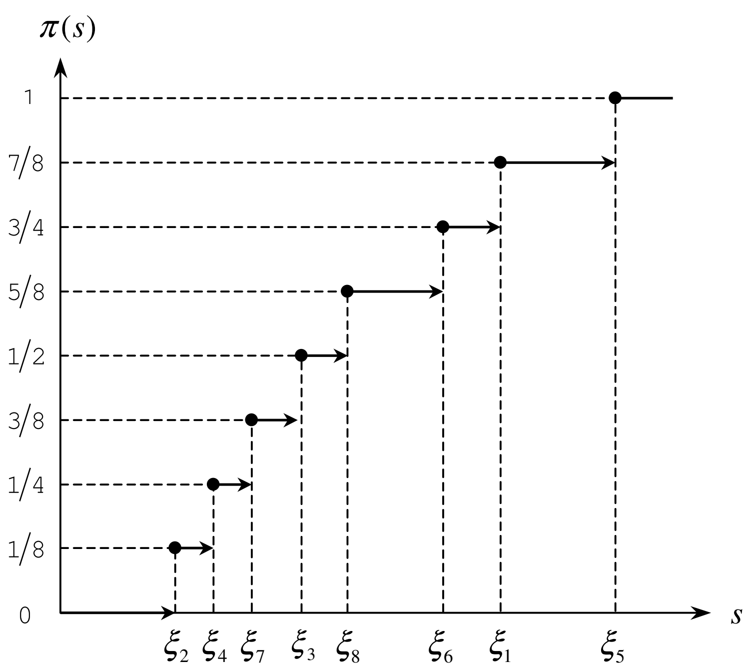

Figure 3: Function $\pi_\xi(s)$.

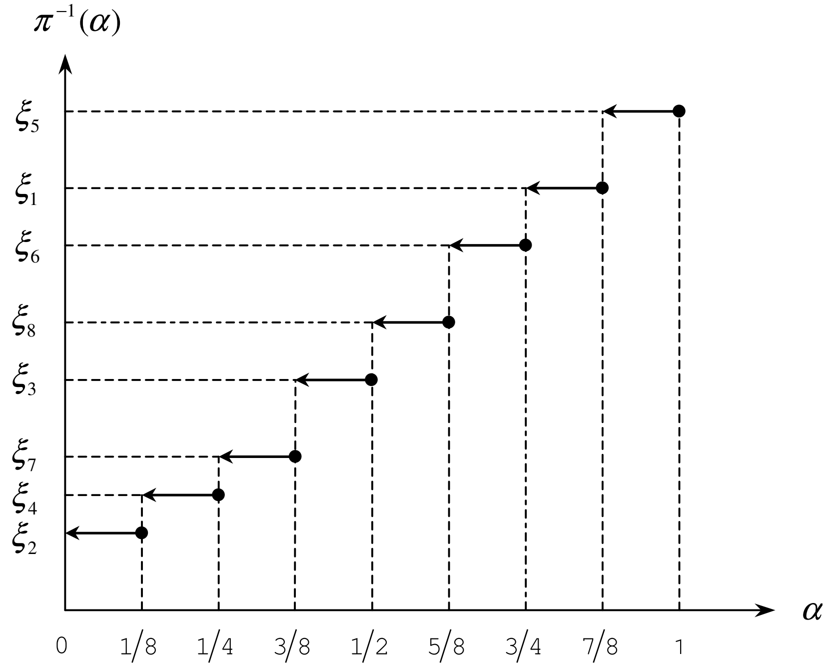

Figure 4: Inverse function $\pi_{\xi}^{-1}(\alpha)$.

where $\Xi_\alpha = \left\{ \xi_k \mid \xi_k > \zeta(\alpha),\ k = \overline{1, N} \right\}$.

Note the first term in the right-hand side of (16) appears due to the inequality $\pi_\xi\left(\pi_\xi^{-1}(\alpha)\right) \geq \alpha$. If $(1-\alpha) * 100\%$ of the worst drawdowns can be counted precisely, then $\pi_\xi\left(\pi_\xi^{-1}(\alpha)\right) = \alpha$ and the first term in the right-hand side of (16) disappears. Definition (16) follows from the framework of the CVaR methodology developed in [18, 19]. Close relation between the CVaR and CV@R is reflected in the following remark.

Remark 3. We would like to emphasize that $\mathrm{CV@R}_\alpha$, given by (16), and functional CVaR_α, presented in [20] (*p. 7, example 4*), are linearly dependent, i.e. if X is an arbitrary random variable then

$$\mathrm{CV@R}_\alpha(X) = \frac{1}{1-\alpha}\left(E(X) + \alpha\ \mathrm{CVaR}_\alpha(X)\right). \tag{17}$$

Thus, use of the CV@R or CVaR is only the matter of convenience.

Definition 4. *In a single scenario case, the CDD with tolerance level* $\alpha \in [0,1]$ *is the CV@R applied to the drawdown functional,* $\mathcal{AD}(w)$*,*

$$\Delta_\alpha(w) = \mathrm{CV@R}_\alpha(\mathcal{AD}(w)). \tag{18}$$

Equivalently, interpreting ξ_k*,* $k = \overline{1, N}$*, to be observations of a "random variable"* ξ*,* α*-CDD is the* $\mathrm{CV@R}_\alpha$ *of a loss function* $\mathcal{AD}(w)$*.*

3.2 Properties of CV@R and CDD

CDD is an example of a functional generalizing properties of deviation measures to a dynamic case. However, since CDD is closely related to CVaR and CVaR properties are studied in detail in [18, 19], it is useful to discuss CDD properties based on properties of CVaR.

Proposition 2. $\mathrm{CV@R}_\alpha(\xi)$ *satisfies the following properties*

1. *Constant translation:* $\mathrm{CV@R}_\alpha(\xi + const) = \mathrm{CV@R}_\alpha(\xi) + const$, $\forall\alpha \in [0,1]$.
2. *Positive homogeneity:* $\mathrm{CV@R}_\alpha(\lambda\,\xi) = \lambda\,\mathrm{CV@R}_\alpha(\xi)$, $\forall\lambda \geq 0$ *and* $\forall\alpha \in [0,1]$.
3. *Monotonicity: if* $\xi_k \leq \eta_k$, $\forall k\overline{1,k}$, *then* $\mathrm{CV@R}_\alpha(\xi) \leq \mathrm{CV@R}_\alpha(\eta)$, $\forall\alpha \in [0,1]$.
4. *Convexity: if* $\xi_\lambda = \lambda\,\xi_a + (1-\lambda)\,\xi_b$ *is a linear combination of any two drawdown sample paths* ξ_a *and* ξ_b *with* $\lambda \in [0,1]$, *then* $\mathrm{CV@R}_\alpha(\xi_\lambda) \leq \lambda\,\mathrm{CV@R}_\alpha(\xi_a) + (1-\lambda)\,\mathrm{CV@R}_\alpha(\xi_b)$.

Proof. Based on linear relation between $\mathrm{CV@R}_\alpha$ and CVaR_α, given by (17), properties $1-4$ are direct consequence of CVaR_α properties discussed in [20]. □

Proposition 3. *The CDD,* $\Delta_\alpha(w)$, *satisfies the properties of deviation measures, i.e.*

1. *Nonnegativity:* $\Delta_\alpha(w) \geq 0$, $\forall\alpha \in [0,1]$.
2. *Insensitivity to constant shift:* $\Delta_\alpha(w + const) = \Delta_\alpha(w)$, $\forall\alpha \in [0,1]$.
3. *Positive homogeneity:* $\Delta_\alpha(\lambda\,w) = \lambda\,\Delta_\alpha(w)$, $\forall\lambda \geq 0$ *and* $\forall\alpha \in [0,1]$.
4. *Convexity: if* $w_\lambda = \lambda\,w_a + (1-\lambda)\,w_b$ *is a linear combination of any two sample paths of uncompounded cumulative rate of returns* w_a *and* w_b *with* $\lambda \in [0,1]$, *then* $\Delta_\alpha(w_\lambda) \leq \lambda\,\Delta_\alpha(w_a) + (1-\lambda)\,\Delta_\alpha(w_b)$.

Proof. Properties $1-4$ follow from Propositions 1 and 2. Indeed, based on the relation between the CDD and CV@R, i.e. $\Delta_\alpha(w) = \mathrm{CV@R}_\alpha(\mathcal{AD}(w))$, the first property is a direct consequence of $\mathcal{AD}(w)$ *nonnegativity.* Properties $2-4$ are proved, respectively,

$$
\begin{aligned}
\Delta_\alpha(w+c) &= \mathrm{CV@R}_\alpha(\mathcal{AD}(w+c)) = \mathrm{CV@R}_\alpha\left(\mathcal{AD}(w)\right) = \Delta_\alpha(w),\\
\Delta_\alpha(\lambda\,w) &= \mathrm{CV@R}_\alpha(\mathcal{AD}(\lambda\,w)) = \mathrm{CV@R}_\alpha(\lambda\,\mathcal{AD}(w)) = \lambda\,\mathrm{CV@R}_\alpha\left(\mathcal{AD}(w)\right) = \lambda\,\Delta_\alpha(w),\\
\Delta_\alpha(w_\lambda) &= \mathrm{CV@R}_\alpha(\mathcal{AD}(\lambda\,w_a + (1-\lambda)w_b)) \leq \mathrm{CV@R}_\alpha\left(\lambda\,\mathcal{AD}(w_a) + (1-\lambda)\mathcal{AD}(w_b)\right)\\
&\leq \lambda\,\mathrm{CV@R}_\alpha\left(\mathcal{AD}(w_a)\right) + (1-\lambda)\mathrm{CV@R}_\alpha\left(\mathcal{AD}(w_b)\right) = \lambda\,\Delta_\alpha(w_a) + (1-\lambda)\Delta_\alpha(w_b).
\end{aligned}
$$

Note the monotonicity property of CV@R is used in the first line of the proof of CDD convexity. □

Proposition 4. *Maximum drawdown (10) and average drawdown (11) are the special cases of the* α*-CDD functional, namely,*

$$
\mathrm{MaxDD}(w) = \Delta_1(w), \quad \mathrm{AvDD}(w) = \Delta_0(w). \tag{19}
$$

Proof. To prove the first formula of (19), we assume that $\zeta(1) < \infty$. Based on this assumption, in the case of $\alpha = 1$, we have $\zeta(1-) = \pi_\xi^{-1}(1-) = \pi_\xi^{-1}(1) = \zeta(1)$, i.e. function $\zeta(\alpha)$ is constant in the left vicinity of 1. Hence, $\pi_\xi(\zeta(1-)) = \pi_\xi(\zeta(1)) = 1$, $\Xi_1 = \emptyset$ and

$$
\Delta_1(w) = \zeta(1)\lim_{\alpha\to 1-}\left(\frac{\pi_\xi\left(\zeta(\alpha)\right) - \alpha}{1-\alpha}\right) = \zeta(1)\lim_{\alpha\to 1-}\left(\frac{1-\alpha}{1-\alpha}\right) = \zeta(1) = \mathrm{MaxDD}(w).
$$

When $\alpha = 0$, according to the definition (13), $\zeta(0) = 0$, $\Xi_0 = \left\{\xi_k \mid k = \overline{1,N}\right\}$ and, consequently,

$$
\Delta_0(w) = \frac{1}{N}\sum_{t_k\in\Xi_0}\xi_k = \frac{1}{N}\sum_{k=1}^{N}\xi_k = \mathrm{AvDD}(w).
$$

□

Theorem 1. $\mathrm{CV@R}_\alpha(\xi)$ *can be presented in the alternative form*

$$
\mathrm{CV@R}_\alpha(\xi) = \frac{1}{1-\alpha}\int_\alpha^1 \pi_\xi^{-1}(q)\,dq, \tag{20}
$$

which is mathematically equivalent to (16).

Proof. Let $\left\{s_j \mid j = \overline{1,J}\right\}$ be the set of the ordered values of ξ_k, $k = \overline{1,N}$, where J is the number of different values of ξ_k, $k = \overline{1,N}$, such that $s_1 < s_2 < \ldots < s_J$ and $n_j \geq 1$ is the multiplicity of s_j, i.e. $n_j = \sum\limits_{k=1}^{N} I_{\{\xi_k = s_j\}}$ and $\sum\limits_{j=1}^{J} n_j = N$. Defining $q_j = \frac{1}{N}\sum\limits_{l=1}^{j} n_l$, step functions π_ξ and π_ξ^{-1} are determined by the set of (s_j, q_j), $j = \overline{1,J}$, i.e.

$$\pi_\xi(s_j) = q_j, \quad \pi_\xi^{-1}(q_j) = s_j. \tag{21}$$

Let $s_0 = 0$ and $q_0 = 0$, then since $\bigcap\limits_{j=1}^{J}(q_{j-1}, q_j] = \emptyset$ and $\bigcup\limits_{j=1}^{J}(q_{j-1}, q_j] = (0,1]$, for any value of $\alpha \in (0,1]$, there exists j^* from $\overline{1,J}$ such that $\alpha \in (q_{j^*-1}, q_{j^*}]$. Using (21) and condition $\alpha \in (q_{j^*-1}, q_{j^*}]$, we obtain

$$\zeta(\alpha) = s_{j^*}, \quad \pi_\xi(\zeta(\alpha)) = q_{j^*},$$

and, consequently,

$$\frac{1}{N}\sum_{t_k \in \Xi_\alpha} \xi_k = \frac{1}{N}\sum_{j=j^*+1}^{J} s_j\, n_j = \sum_{j=j^*+1}^{J} \pi_\xi^{-1}(q_j)\,(q_j - q_{j-1}) = \int\limits_{q_{j^*}}^{1} \pi_\xi^{-1}(q)\, dq.$$

Accounting for the last relations, for any $\alpha \in (0,1)$, the integral in the right-hand side of (20) is presented

$$\int\limits_{\alpha}^{1} \pi_\xi^{-1}(q)\, dq = (q_{j^*} - \alpha)\, s_{j^*} + \int\limits_{q_{j^*}}^{1} \pi_\xi^{-1}(q)\, dq = (\pi_\xi(\zeta(\alpha)) - \alpha)\,\zeta(\alpha) + \frac{1}{N}\sum_{t_k \in \Xi_\alpha} \xi_k,$$

which coincides with the expression (16) with accuracy of multiplier $(1-\alpha)^{-1}$.

There are only two cases left to consider, namely, when $\alpha = 0$ and $\alpha = 1$. Assuming $\pi_\xi^{-1}(1) < \infty$, we have, respectively,

$$\begin{aligned}
\Delta_0(w) &= \int\limits_0^1 \pi_\xi^{-1}(q)\, dq = \tfrac{1}{N}\sum_{j=1}^{J} n_j\, s_j = \tfrac{1}{N}\sum_{k=1}^{N} \xi_k = \mathrm{AvDD}(w),\\
\Delta_1(w) &= \lim_{\alpha \to 1}\left(\tfrac{1}{1-\alpha}\int\limits_\alpha^1 \pi_\xi^{-1}(q)\, dq\right) = \pi_\xi^{-1}(1) = \mathrm{MaxDD}(w).
\end{aligned}$$

□

Remark 4. Let X be an arbitrary random variable with the cumulative distribution function $F_X(t) = \Pr\{X \leq t\}$. Assuming $F_X^{-1}(\alpha)$ to be the inverse function of $F_X(t)$, functionals $\mathrm{CV@R}_\alpha$ and CVaR_α are expressed, respectively,

$$\mathrm{CV@R}_\alpha(X) = \frac{1}{1-\alpha}\int\limits_\alpha^1 F_X^{-1}(q)\, dq, \quad \mathrm{CVaR}_\alpha(X) = -\frac{1}{\alpha}\int\limits_0^\alpha F_X^{-1}(q)\, dq. \tag{22}$$

Relation (17) can be verified based on (22). A reader interested in CVaR methodology may refer to [18, 19, 20].

Example 1. To illustrate the concept of the CV@R, let us calculate $\mathrm{CV@R}_{0.7}(\xi)$ for drawdown time series ξ shown on Figure 2. According to Figure 4, $\zeta(0.7) = \pi_\xi^{-1}(0.7) = \xi_6$, and, consequently, from Figure 3, $\pi_\xi(\zeta(0.7) = \pi_\xi(\xi_6) = 0.75$. Using formula (16), we obtain $\mathrm{CV@R}_{0.7}(\xi) = \frac{(0.75-0.7)}{1-0.7}\,\xi_6 + \frac{1}{1-0.7}\,\frac{(\xi_1+\xi_5)}{8} = \frac{1}{6}\,\xi_6 + \frac{5}{12}\,(\xi_1+\xi_5)$. To verify this result, we can calculate $\mathrm{CV@R}_{0.7}(\xi)$ based on (20). Namely, following Figure 4, we have $\mathrm{CV@R}_{0.7}(\xi) = \frac{1}{1-0.7}\,((0.75-0.7)\,\xi_6 + (0.875-0.75)\,\xi_1 + (1-0.875)\,\xi_5) = \frac{1}{6}\,\xi_6 + \frac{5}{12}\,\xi_1 + \frac{5}{12}\,\xi_5$.

Example 2. For the drawdown time series shown on Figure 2, $\mathrm{MaxDD}(w) = \xi_5$ and $\mathrm{AvDD}(w) = \frac{1}{8}\sum\limits_{k=1}^{8}\xi_k$.

3.3 Mixed Conditional Drawdown

The notion of Conditional Drawdown can be generalized by considering convex combinations of Conditional Drawdowns corresponding to different confidence levels. This idea is essentially based on risk profiling, i.e. assignment of specific weights for CDDs with predetermined confidence levels.

Definition 5. *Given a risk profile $\chi(\alpha)$ such that*

1) $d\chi(\alpha) \geq 0$;

2) $\int\limits_0^1 d\chi(\alpha) = 1$;

mixed CDD, is defined

$$\Delta_\chi^+(w) = \int\limits_0^1 \Delta_\alpha(w)\; d\chi(\alpha). \tag{23}$$

Obviously, the mixed CDD preserves all properties of $\Delta_\alpha(w)$ stated in proposition 4. A fund manager can flexibly express his or her risk preferences by shaping $\chi(\alpha)$.

Proposition 5. *The mixed CDD can be presented in the alternative form*

$$\Delta_\chi^+(w) = \int\limits_0^1 \pi_\xi^{-1}(\alpha)\; \mu(\alpha)\; d\alpha, \tag{24}$$

with "spectrum" $\mu(\alpha)$ to be:

1) nonnegative on $[0,1]$;

2) nondecreasing on $[0,1]$;

3) $\int\limits_0^1 \mu(\alpha)\; d\alpha = 1$.

The relation between $\chi(\alpha)$ in (23) and $\mu(\alpha)$ in (24) is

$$d\mu(\alpha) = \frac{1}{1-\alpha}\, d\chi(\alpha).$$

Proof. Expressing $\Delta_\alpha(w)$ in the form of (20), consider

$$\begin{aligned}\Delta_\chi^+(w) &= \int_0^1 \left(\frac{1}{1-\alpha} \int_\alpha^1 \pi_\xi^{-1}(q)\, dq \right) d\chi(\alpha) = \int_0^1 \left(\frac{1}{1-\alpha} \int_0^1 \pi_\xi^{-1}(q)\, I_{\{q\ge\alpha\}}\, dq \right) d\chi(\alpha) \\ &= \int_0^1 \pi_\xi^{-1}(q) \left(\int_0^1 \frac{1}{1-\alpha} I_{\{q\ge\alpha\}}\, d\chi(\alpha) \right) dq = \int_0^1 \pi_\xi^{-1}(q) \left(\int_0^q \frac{1}{1-\alpha}\, d\chi(\alpha) \right) dq = \int_0^1 \pi_\xi^{-1}(q)\, \mu(q)\, dq,\end{aligned}$$

where $\mu(\alpha) = \int_0^\alpha \frac{1}{1-q}\, d\chi(q)$ satisfies all properties *1) – 3)*. Indeed, $\mu(\alpha)$ is nonnegative and nondecreasing, since $d\mu(\alpha) = \frac{1}{1-\alpha}\, d\chi(\alpha) \geq 0$. Moreover, $\int_0^1 \mu(\alpha)\, d\alpha = \int_0^1 \int_0^1 \frac{1}{1-q} I_{\{\alpha\ge q\}}\, d\chi(q)\, d\alpha = 1$. Obviously, conditions *1) – 3)* are necessarily satisfied by function $\mu(\alpha)$, since they are derived from the properties of function $\chi(\alpha)$. However, if function $\mu(\alpha)$ satisfies conditions *1) – 3)* then it is sufficient for (24) to be *constant translating, positively homogeneous, monotonic and convex* with respect to ξ. The last fact comes from a direct verification of those properties. □

Corollary 1. The non-decrease property of "spectrum," $\mu(\alpha)$, is a necessary condition for the mixed CDD to be convex. This property has an obvious but important interpretation, namely, *the greater drawdown quantile,* π_ξ^{-1}*, is, the greater penalty coefficient,* μ*, should be assigned.* A similar conclusion regarding risk spectrum in coherent risk measures has been drawn in [1]. This conclusion is a consequence of a general *coherency principle*, stating: *the greater risk is, the more it should be penalized*, see [2].

Example 3. MaxDD and AvDD are mixed CDDs with risk profiles $\chi(\alpha) = I_{\{\alpha\ge1\}}$ and $\chi(\alpha) = I_{\{\alpha>0\}}$, respectively.

Discrete risk profile. An important case is when risk profile, $\chi(\alpha)$, is specified by the discrete set of points $\chi_i = d\chi(\alpha_i)$, $i = \overline{1, L}$. In this case, the mixed CDD is expressed

$$\Delta_\chi^+(w) = \sum_{i=1}^L \chi_i\, \Delta_{\alpha_i}(w), \tag{25}$$

where $\sum_{i=1}^L \chi_i = 1$ and $\chi_i \geq 0$. Consequently, "spectrum" function is presented by

$$\mu(\alpha) = \sum_{i=1}^L \frac{\chi_i}{1-\alpha_i} I_{\{\alpha\ge\alpha_i\}}. \tag{26}$$

Detail. Interchanging summation and integration operations in $\Delta_\chi^+(w)$, the result follows

$$\Delta_\chi^+(w) = \sum_{i=1}^L \chi_i\, \Delta_{\alpha_i}(w) = \sum_{i=1}^L \frac{\chi_i}{1-\alpha_i} \int_{\alpha_i}^1 \pi_\xi^{-1}(q)\, dq = \int_0^1 \left(\sum_{i=1}^L \frac{\chi_i}{1-\alpha_i} I_{\{\alpha\ge\alpha_i\}} \right) \pi_\xi^{-1}(q)\, dq.$$

Obviously, (26) is a positive nondecreasing function.

4 Optimization techniques for Conditional Drawdown computation

This section develops optimization techniques for CDD efficient computation.

Formulas (16) and (20) require to calculate the value of $\zeta(\alpha)$ first, which doubles the calculation process. However, there is an optimization procedure allowing one to obtain the values of threshold $\zeta(\alpha)$ and CDD simultaneously. This procedure is especially important in a large scale optimization.

4.1 Computation of the CDD for a given drawdown time series

In the case when a time series of drawdowns is given, computation of the α-CDD is reduced to computation of $\mathrm{CV@R}_\alpha(\xi)$.

Theorem 2. *Given a time series of instrument's drawdowns $\xi = (\xi_1, \ldots, \xi_N)$, corresponding to time moments $\{t_1, \ldots, t_N\}$, the CDD functional is presented by $\mathrm{CV@R}_\alpha(\xi)$, which computation is reduced to the following linear programming procedure*

$$\begin{aligned} \mathrm{CV@R}_\alpha(\xi) = \quad & \min_{y,\,z} \quad y + \tfrac{1}{(1-\alpha)N} \sum_{k=1}^{N} z_k \\ & s.\,t. \quad z_k \geq \xi_k - y, \quad z_k \geq 0,\ k = \overline{1, N}, \end{aligned} \tag{27}$$

leading to a single optimal value of y equal to $\zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) > \alpha$, and to a closed interval of optimal y with the left endpoint of $\zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) = \alpha$.

Proof. Let us introduce a piece-wise function

$$h(y) = y + \frac{1}{(1-\alpha)N} \sum_{k=1}^{N} [\xi_k - y]^+, \tag{28}$$

where $[\xi_k - y]^+ = \max\{\xi_k - y,\ 0\}$, and establish the following relation

$$\mathrm{CV@R}_\alpha(\xi) = \min_y\ h(y). \tag{29}$$

The derivative of $h(y)$ with respect to y is presented

$$\frac{d}{dy} h(y) = 1 - \frac{1}{(1-\alpha)N} \sum_{k=1}^{N} I_{\{y<\xi_k\}} = 1 - \frac{1}{(1-\alpha)N} \sum_{k=1}^{N} \left(1 - I_{\{\xi_k \leq y\}}\right) = \frac{\pi_\xi(y) - \alpha}{1-\alpha}. \tag{30}$$

Note $\frac{d}{dy} h(y)$ is continuous for all values of y, except the set of points $y = \left\{ \xi_k \mid k = \overline{1, N} \right\}$. The necessary condition for function $h(y)$ to attain an extremum is

$$\frac{d^-}{dy} h(y) \leq 0 \leq \frac{d^+}{dy} h(y), \tag{31}$$

where $\frac{d^-}{dy} h(y) = \frac{1}{(1-\alpha)} (\pi_\xi(y - 0) - \alpha)$ and $\frac{d^+}{dy} h(y) = \frac{1}{(1-\alpha)} (\pi_\xi(y + 0) - \alpha)$ are left and right derivatives, respectively, which coincide with each other for all y except $y = \left\{ \xi_k \mid k = \overline{1, N} \right\}$. According to (30) and (31), an optimal value y^* should satisfy inequalities

$$\pi_\xi(y^* - 0) \leq \alpha \leq \pi_\xi(y^* + 0),$$

which have a unique solution $y^* = \zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) > \alpha$ (see Remark 2), i.e. if $y^* \neq \left\{ \xi_k \mid k = \overline{1, N} \right\}$. However, if $\pi_\xi(\zeta(\alpha)) = \alpha$, then there is a closed interval of optimal values y^*, with the left endpoint of $\zeta(\alpha)$, namely, $y^* \in [\zeta(\alpha), \zeta(\alpha + 0)]$, where $\pi_\xi(\zeta(\alpha + 0)) > \alpha$. Hence, two cases are considered:

a) $y^* = \zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) > \alpha$;

b) $y^* \in [\zeta(\alpha), \zeta(\alpha + 0)]$ if $\pi_\xi(\zeta(\alpha)) = \alpha$.

In both cases, equality $[\xi_k - y^*]^+ = (\xi_k - y^*)\, I_{\{\xi_k \geq y^*\}} = (\xi_k - y^*)\, I_{\{\xi_k > \zeta(\alpha)\}}$ holds with respect to all ξ_k, $k = \overline{1, N}$, for any fixed y^*. Thus, based on this fact, we obtain

$$
\begin{aligned}
\min_y h(y) &= h(y^*) = y^* + \tfrac{1}{(1-\alpha)N} \sum_{k=1}^{N} [\xi_k - y^*]^+ \\
&= \tfrac{1}{1-\alpha} \left(1 - \alpha - \tfrac{1}{N} \sum_{k=1}^{N} I_{\{\xi_k > \zeta(\alpha)\}} \right) y^* + \tfrac{1}{(1-\alpha)N} \sum_{k=1}^{N} \xi_k I_{\{\xi_k > \zeta(\alpha)\}} \\
&= \tfrac{(\pi_\xi(\zeta(\alpha)) - \alpha)}{1-\alpha} y^* + \tfrac{1}{(1-\alpha)N} \sum_{t_k \in \Xi_\alpha} \xi_k,
\end{aligned}
$$

where $\frac{(\pi_\xi(\zeta(\alpha))-\alpha)}{1-\alpha} y^* = \frac{(\pi_\xi(\zeta(\alpha))-\alpha)}{1-\alpha} \zeta(\alpha)$ in the case of *a)*, and $\frac{(\pi_\xi(\zeta(\alpha))-\alpha)}{1-\alpha} y^* = 0$ in the case of *b)*. Consequently, $\min\limits_y h(y)$ coincides with the definition of the CDD.

Since expression $\sum\limits_{k=1}^{N} [\xi_k - y]^+$ is minimized, it can equivalently be presented by the sum of nonnegative auxiliary variables $z_k \geq 0$, $k = \overline{1, N}$, with imposing additional constraints $z_k \geq \xi_k - y$, $k = \overline{1, N}$. □

Corollary 2. $\mathrm{CV@R}_\alpha(\xi)$ *is an optimal value for the objective function of the following knapsack problem*

$$
\begin{aligned}
\mathrm{CV@R}_\alpha(\xi) = \quad & \max_q \sum_{k=1}^{N} \xi_k \, q_k \\
& s.\,t. \sum_{k=1}^{N} q_k = 1, \;\; 0 \leq q_k \leq \tfrac{1}{(1-\alpha)N}, \;\; k = \overline{1, N}.
\end{aligned} \tag{32}
$$

The value of $\mathrm{CV@R}_\alpha(\xi)$ *can be found in* $O(n \log_2 n)$ *time.*

Proof. It is enough to observe that knapsack problem (32) is *dual* to linear programming problem (27). Based on duality theory, optimal values of the objective functions in (27) and (32) should coincide.

Problem (32) can be solved by the standard *greedy* algorithm in $O(n \log_2 n)$ time. The algorithm sorts items according to their *"costs"* $\left\{ \xi_k \,|\, k = \overline{1, N} \right\}$. Let $\lfloor a \rfloor$ denote the integer part of real number a. Obviously, q-variables, corresponding to the largest $\lfloor (1-\alpha)N \rfloor$ "costs," have optimal values equal to $\frac{1}{(1-\alpha)N}$, and the q-variable, corresponding to the $(\lfloor (1-\alpha)N \rfloor + 1)^{th}$ "cost" in the sorted order, has optimal value equal to $1 - \frac{\lfloor (1-\alpha)N \rfloor}{(1-\alpha)N}$. The rest of q-variables equal 0. In this case, the complexity of the algorithm is mainly determined by a sorting procedure, which, in this case, requires at least $O(n \log_2 n)$ operations. □

Formulation (32) is closely related to the presentation of CV@R based on the concept of a *risk envelope*, which is a closed, convex set of probabilities containing 1. A reader interested in this concept and related material may refer to [20] and [21].

4.2 Computation of the CDD for a given sample path of instrument's rates of returns

Suppose a sample path of instrument's rates of returns $(r_1, \ldots, r_N)$, corresponding to time moments $\{t_1, \ldots, t_N\}$, is given. In this case, uncompounded cumulative instrument's rate of return at time moment t_k is $w_k = \sum\limits_{l=1}^{k} r_l$, and the CDD is presented in the form of $\Delta_\alpha(w)$.

Proposition 6. *Given a sample path of instrument's rates of returns* $(r_1, \ldots, r_N)$*, the CDD functional,* $\Delta_\alpha(w)$*, is computed by the following optimization procedure*

$$\begin{array}{rll}\Delta_\alpha(w) = & \min\limits_{u,\,y,\,z} & y + \frac{1}{(1-\alpha)N} \sum\limits_{k=1}^{N} z_k \\ & s.\,t. & z_k \geq u_k - y, \\ & & u_k \geq u_{k-1} - r_k,\ u_0 = 0, \\ & & z_k \geq 0,\ u_k \geq 0,\ k = \overline{1,N},\end{array} \tag{33}$$

which leads to a single optimal value of y equal to $\zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) > \alpha$, and to a closed interval of optimal y with the left endpoint of $\zeta(\alpha)$ if $\pi_\xi(\zeta(\alpha)) = \alpha$.

Proof. Due to the relation $\Delta_\alpha(w) = \mathrm{CV@R}_\alpha(\mathcal{AD}(w)) = \mathrm{CV@R}_\alpha(\xi)$, optimization problem (33) is a direct consequence of (27). With using recursive formula $\xi_k = [\xi_{k-1} - r_k]^+$, constraint $z_k \geq \xi_k - y$ in (27) is reduced to $z_k \geq u_k - y$, where nonnegative auxiliary variables u_k satisfy additional constraints $u_k \geq \xi_{k-1} - r_k$, $k = \overline{1,N}$, with $u_0 = 0$. □

Corollary 3. *Given a sample path of instrument's rates of returns $(r_1, \ldots, r_N)$, the CDD functional, $\Delta_\alpha(w)$, is computed by the following optimization procedure*

$$\begin{array}{rll}\Delta_\alpha(w) = & \max\limits_{q,\,\eta} & -\sum\limits_{k=1}^{N} r_k\, \eta_k \\ & s.\,t. & \sum\limits_{k=1}^{N} q_k = 1,\ \ \eta_k - \eta_{k+1} \leq q_k \leq \frac{1}{(1-\alpha)N}, \\ & & q_k \geq 0,\ \ \eta_k \geq 0,\ \ \eta_{N+1} = 0,\ \ k = \overline{1,N}.\end{array} \tag{34}$$

Proof. Problem (34) is *dual* to linear programming program (32). □

4.3 Computation of the mixed CDD

Theorem 2 and all its corollaries can be easily extended to computation of the mixed CDD. For illustration purpose, we extend corollary # only.

Proposition 7. *Given a sample path of instrument's rates of returns $\left\{r_k \,|\, k = \overline{1,N}\right\}$ and discrete risk profile $\chi_i = d\chi(\alpha_i)$, $i = \overline{1,L}$, the mixed CDD, $\Delta^+_\chi(w)$, is computed by the following optimization procedure*

$$\begin{array}{rll}\Delta^+_\chi(w) = & \min\limits_{u,\,y,\,z} & \sum\limits_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha)N} \sum\limits_{k=1}^{N} z_{ik} \right) \\ & s.\,t. & z_{ik} \geq u_k - y_i, \\ & & u_k \geq u_{k-1} - r_k,\ u_0 = 0, \\ & & z_{ik} \geq 0,\ u_k \geq 0,\ i = \overline{1,L},\ k = \overline{1,N}.\end{array} \tag{35}$$

Proof. Formulation (35) is a direct consequence of mixed CDD definition (25) and optimization problem (33). Notice that auxiliary variables u_k do not have index i, since they determine the drawdown sequence that is the same for all α_i. □

5 "Multi-scenario" CV@R and Drawdown measure

This section presents concept of the "Multi-scenario" CV@R and Drawdown measure, which, in fact, are the CV@R and CDD defined in the case of several sample paths for uncompounded cumulative portfolio rate of return. Here, the main results obtained for the CDD in the case of a single sample path are extended to the case of several sample paths.

Let Ω denote a discrete set of random events, i.e. $\Omega = \{\, \omega_j \mid j = \overline{1,K} \}$, and let p_j be the probability of event ω_j. Certainly, $\forall j : p_j \geq 0$, and $\sum\limits_{j=1}^{K} p_j = 1$. Suppose $r_j(t_k) = (r_{1j}(t_k),\, r_{2j}(t_k),\, \ldots,\, r_{mj}(t_k))$, $k = \overline{1,N}$, is the j^{th} sample path for the random vector of risky assets' rates of returns, corresponding to random event $\omega_j \in \Omega$ and time interval $[0,T]$ presented by the discrete set of time moments $\{t_0 = 0,\, t_1,\, t_2,\, \ldots,\, t_N = T\}$. Consequently, the j^{th} sample path for the rate of return and uncompounded cumulative rate of return of a portfolio with capital weights $x(t_k) = (x_0(t_k),\, x_1(t_k),\, x_2(t_k),\, \ldots,\, x_m(t_k))$ are defined respectively

$$r_{jk}^{(p)}(x(t_k)) = r_j(t_k) \cdot x(t_k) = \sum_{i=1}^{m} r_{ij}(t_k)\, x_i(t_k), \tag{36}$$

$$w_{jk}(x(t_k)) = \begin{cases} 0, \quad k = 0, \\ \sum\limits_{l=1}^{k} r_{jl}^{(p)}(x(t_l)), \quad k = \overline{1,N}. \end{cases} \tag{37}$$

To simplify notations, we use w_{jk} instead of $w_{jk}(x(t_k))$ implying that w_{jk} is always a function of $x(t_k)$. In a multi-scenario case, w denotes matrix $\{w_{jk}\}$, $j = \overline{1,K}$, $k = \overline{0,N}$.

5.1 "Multi-scenario" CV@R

Definition 6. *In a multi-scenario case, the $\mathcal{AD}(w)$ is a matrix-functional defined on $\Omega \times [0,T]$*

$$\mathcal{AD}(w) = \xi = \{\xi_{jk}\}, \quad \xi_{jk} = \max_{0 \leq l \leq k} \{w_{jl}\} - w_{jk}, \quad j = \overline{1,K},\ k = \overline{1,N}. \tag{38}$$

All $\mathcal{AD}$ properties stated in Proposition 1 hold in a multi-scenario case. Indeed, based on (38), properties $1-4$ in Proposition 1 can be verified directly. Matrix $\mathcal{AD}(w)$ is interpreted to be "drawdown surface" ξ_{jk}, $(\omega_j, t_k) \in \Omega \times [0,T]$.

Definition 7. *Similar to definitions of MaxDD and AvDD in single scenario case, MaxDD and AvDD are defined on $\Omega \times [0,T]$, respectively*

$$\text{MaxDD}(w) = \max_{1 \leq j \leq K,\ 1 \leq k \leq N} \{\xi_{jk}\}, \tag{39}$$

$$\text{AvDD}(w) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j\, \xi_{jk}. \tag{40}$$

Definition 8. *Indicator function for "drawdown surface," its inverse function and threshold plane, $\zeta(\alpha)$, are defined, respectively*

$$\pi_\xi(s) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j\, I_{\{\xi_{jk} \leq s\}}, \tag{41}$$

$$\pi_\xi^{-1}(\alpha) = \begin{cases} \inf \{\, s \mid \pi_\xi(s) \geq \alpha \}, \quad \alpha \in (0,1], \\ 0, \quad \alpha = 0, \end{cases} \tag{42}$$

$$\zeta(\alpha) = \pi_\xi^{-1}(\alpha). \tag{43}$$

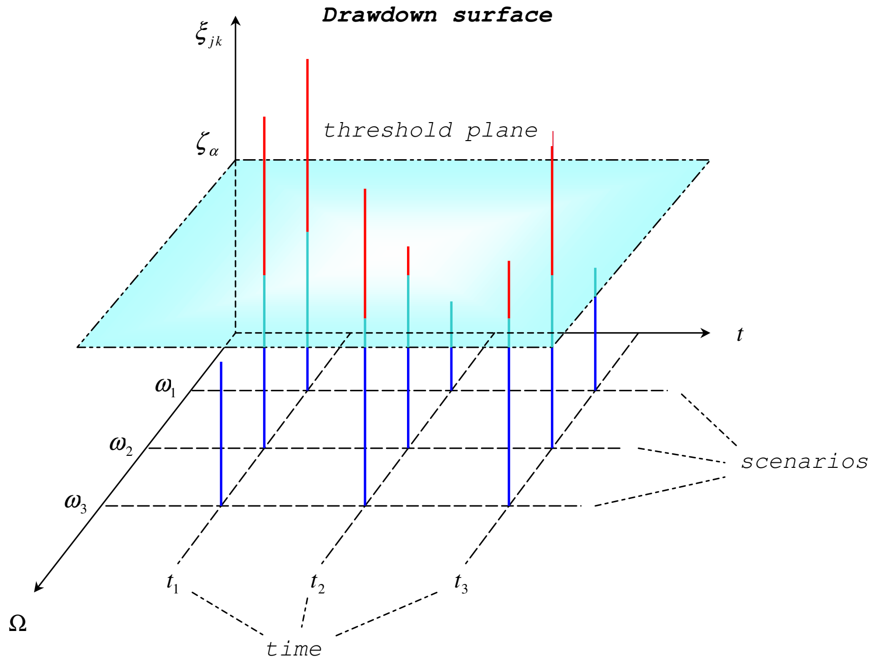

Figure 5: "Drawdown surface" ξ_{jk} and threshold plane $\zeta(\alpha)$.

Figure 5 illustrates "drawdown surface" ξ_{jk} and threshold plane $\zeta(\alpha)$.

Definition 9. *"Multi-scenario" CV@R may be defined similar to "a single period" CV@R, namely,*

$$\mathrm{CV@R}(\xi) = \left(\frac{\pi_\xi\left(\zeta(\alpha)\right) - \alpha}{1-\alpha} \right) \zeta(\alpha) + \frac{1}{(1-\alpha)N} \sum_{\xi_{jk} \in \, \Xi_\alpha} p_j \, \xi_{jk}, \tag{44}$$

where $\Xi_\alpha = \left\{ \xi_{jk} \mid \xi_{jk} > \zeta(\alpha), \; k = \overline{1, N} \right\}$.

Proposition 8. *"Multi-scenario" CV@R, given by (44), can be presented in the alternative form*

$$\mathrm{CV@R}(\xi) = \frac{1}{1-\alpha} \int_\alpha^1 \pi_\xi^{-1}(q)\, dq, \tag{45}$$

where $\pi_\xi^{-1}(q)$ *is the inverse function given by (42).*

Proof. Similar to the proof of Theorem 1. □

Remark 5. Let X be an arbitrary random variable. Suppose we are given K sample paths $X(t_k, \omega_j)$, $k = \overline{1, N}$, corresponding to random events $\omega_j \in \Omega$ with probabilities p_j such that $\sum_{j=1}^{K} p_j = 1$. Defining an indicator function for X to be $\pi_X(s) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j I_{\{X(t_k, \omega_j) \le s\}}$ (where the inverse function π_X^{-1} is defined similar to (42)), *"multi-scenario"* CV@R may be determined similar to *"a single period"* CV@R, namely, $\mathrm{CV@R}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \pi_X^{-1}(q)\, dq$.

5.2 Drawdown measure

Similar to the case of a single sample path, CDD with tolerance level α is:

a) the average of the worst $(1-\alpha)*100\%$ *drawdowns on "drawdown surface," if the worst* $(1-\alpha)*100\%$ *drawdowns can be counted precisely;*

b) the linear combination of $\zeta(\alpha)$ *and the average of the drawdowns strictly exceeding threshold plane* $\zeta(\alpha)$*, if we are unable to precisely count of* $(1-\alpha)*100\%$ *drawdowns.*

A strict mathematical definition of the drawdown measure is given below.

Definition 10. *In a multi-scenario case, the CDD, with tolerance level* $\alpha \in [0,1]$*, is the "multi-scenario" CV@R*$_\alpha$ *applied to "drawdown surface,"* $\mathcal{AD}(w)$*,*

$$\Delta_\alpha(w) = \mathrm{CV@R}_\alpha(\mathcal{AD}(w)), \tag{46}$$

and drawdown measure is the mixed CDD with risk profile $\chi(\alpha)$

$$\Delta_\chi^+(w) = \int_0^1 \Delta_\alpha(w)\, d\chi(\alpha), \tag{47}$$

where $\Delta_\alpha(w)$ *is given by (46).*

Proposition 9. *Defining matrix operations:* $w + const = \{w_{jk} + const\}$ *and* $\lambda w = \{\lambda w_{jk}\}$*, drawdown measure* $\Delta_\chi^+(w)$ *satisfies the following properties*

1. *Nonnegativity:* $\Delta_\chi^+(w) \geq 0$, $\forall \alpha \in [0,1]$.
2. *Insensitivity to constant shift:* $\Delta_\chi^+(w + const) = \Delta_\chi^+(w)$, $\forall \alpha \in [0,1]$.
3. *Positive homogeneity:* $\Delta_\chi^+(\lambda\, w) = \lambda\, \Delta_\chi^+(w)$, $\forall \lambda \geq 0$ *and* $\forall \alpha \in [0,1]$.
4. *Convexity: if* $w_\lambda = \lambda\, w_1 + (1-\lambda)\, w_2$ *is a linear combination of any* w_1 *and* w_2 *with* $\lambda \in [0,1]$*, then* $\Delta_\chi^+(w_\lambda) \leq \lambda\, \Delta_\chi^+(w_1) + (1-\lambda)\, \Delta_\chi^+(w_2)$.

Proof. Properties 1 – 4 are direct generalization of CDD properties stated in Proposition 4. ◻

Proposition 10. *In the case of discretely determined risk profile, drawdown measure can be computed by the following optimization procedure*

$$\begin{array}{rll} \Delta_\chi^+(w) = & \min\limits_{u,\,y,\,z} & \sum\limits_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha_i)N} \sum\limits_{k=1}^{N} \sum\limits_{j=1}^{K} p_j\, z_{ijk} \right) \\ & s.\,t. & z_{ijk} \geq u_{jk} - y_i, \\ & & u_{jk} \geq u_{j(k-1)} - r_{jk}^{(p)}, \\ & & u_{jk} \geq 0,\ u_{j0} = 0,\ z_{ijk} \geq 0, \\ & & i = \overline{1,L},\ j = \overline{1,K},\ k = \overline{1,N}. \end{array} \tag{48}$$

Proof. Introducing intermediate optimization problems

$$\sum_{i=1}^{L} \chi_i\, \mathrm{CV@R}_{\alpha_i}(\xi) = \min_{y_i} \sum_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha_i)N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j\, [\xi_{jk} - y_i]^+ \right),$$

$$\begin{array}{rll} \sum\limits_{i=1}^{L} \chi_i\, \mathrm{CV@R}_{\alpha_i}(\xi) = & \min\limits_{y_i,\, z_{ijk}} & \sum\limits_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha_i)N} \sum\limits_{k=1}^{N} \sum\limits_{j=1}^{K} p_j\, z_{ijk} \right) \\ & s.\,t. & z_{ijk} \geq \xi_{jk} - y_i,\ z_{ijk} \geq 0, \\ & & i = \overline{1,L},\ j = \overline{1,K},\ k = \overline{1,N}, \end{array}$$

the proof is conducted similar to the proof of Theorem 2. □

6 Portfolio optimization with drawdown measure

In this section we formulate a portfolio optimization problem with using drawdown risk measure and suggest an optimization technique for its solving.

The following assumptions are made:

- *portfolio optimization is based on generation of sample paths for the assets' rates of returns;*
- *portfolio optimization uses uncompounded cumulative portfolio rate of return w rather than compounded portfolio rate of return W.*

Portfolio optimization problem is to maximize the expected value of uncompounded cumulative portfolio rate of return at the final time moment $t_N = T$ subject to a constraint on drawdown measure, i.e.

$$\begin{array}{ll} \max\limits_{x \in X} & E_\omega\left(w(T, \omega, x)\right) = \sum\limits_{j=1}^{K} p_j\, w_{jN}(x) \\ s.\,t. & \Delta_\chi^+(w(x)) \le \gamma, \end{array} \tag{49}$$

where X is the set of linear "technological" constraints and $\gamma \in [0, 1]$ is a proportion of the initial capital allowed to loose.

Contrary to the work of [7, 12], where the vector of portfolio weights was variable within $[0, T]$, we assume portfolio weights $x(t_k)$ to be constant for all t_k, $k = \overline{0, N}$. This special strategy can be achieved by portfolio rebalancing at every t_k, $k = \overline{0, N}$. Justification of this assumption depends on a particular case study.

Based on the assumption made, uncompounded cumulative portfolio rate of return w is rewritten

$$w_{jk}(x) = \sum_{l=1}^{k} r_{jl}^{(p)}(x) = \sum_{i=1}^{m} \sum_{l=1}^{k} r_{ij}(t_l)\, x_i. \tag{50}$$

6.1 Reduction of optimization problem to linear programming (LP) problem

Theorem 3. *Problem (49) is reduced to linear programming (LP) problem*

$$\begin{array}{ll} \max\limits_{u,\, x \in X,\, y,\, z} & \sum\limits_{j=1}^{K} p_j\; w_{jN}(x) \\ s.\,t. & \sum\limits_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha_i)N} \sum\limits_{k=1}^{N} \sum\limits_{j=1}^{K} p_j\, z_{ijk} \right) \le \gamma, \\ & z_{ijk} \ge u_{jk} - y_i, \\ & u_{jk} \ge u_{j(k-1)} - r_{jk}^{(p)}(x), \\ & u_{jk} \ge 0,\ u_{j0} = 0,\ z_{ijk} \ge 0, \\ & i = \overline{1, L},\ j = \overline{1, K},\ k = \overline{1, N}, \end{array} \tag{51}$$

where u_{jk}, y_i and z_{ijk} are auxiliary variables.

Proof. Let piece-wise function $H(x, y)$ be introduced

$$H(x,y) = \sum_{i=1}^{L} \chi_i \left(y_i + \frac{1}{(1-\alpha_i)N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j \, [\xi_{jk}(x) - y_i]^+ \right). \tag{52}$$

According to Proposition 10, Drawdown measure may be presented by

$$\Delta_\chi^+(w(x)) = \sum_{i=1}^{L} \chi_i \, \mathrm{CV@R}_{\alpha_i}(\xi(x)) = \min_y \; H(x,y). \tag{53}$$

Consequently, problem (49) is reduced to

$$\begin{array}{ll} \max\limits_{x\in X} & \sum\limits_{j=1}^{K} p_j \, w_{jN}(x) \\ s.\,t. & \min\limits_y \; H(x,y) \le \gamma, \end{array} \tag{54}$$

The key point of the proof is to show that minimum in the constraint of (54) may be relaxed, i.e to show that problem (54) is equivalent to

$$\begin{array}{ll} \max\limits_{x\in X,\, y} & \sum\limits_{j=1}^{K} p_j \, w_{jN}(x) \\ s.\,t. & H(x,y) \le \gamma, \end{array} \tag{55}$$

The proof of this fact is conducted by relaxing constraint $\min\limits_y H(x,y) \le C\gamma$ in (54), namely, problem (54) is equivalently rewritten

$$\min_{\lambda\ge 0} \max_{x\in X} \left(\sum_{j=1}^{K} p_j \, w_{jN}(x) + \lambda \left(\gamma - \min_y \; H(x,y) \right) \right),$$

$$\min_{\lambda\ge 0} \max_{x\in X,\, y} \left(\sum_{j=1}^{K} p_j \, w_{jN}(x) + \lambda \left(\gamma - H(x,y) \right) \right). \tag{56}$$

However, problem (56) is the *Lagrange relaxation* of (55). Hence, (55) is equivalent to (54). According to Theorem 4 and Proposition 10, LP (51) is a direct consequence of (55). □

Corollary 4. *In the cases of* MaxDD(w) *and* AvDD(w), *corresponding to mixed CDD with risk profiles of* $\chi(\alpha) = I_{\{\alpha>0\}}$ *and* $\chi(\alpha) = I_{\{\alpha\ge 1\}}$, *LP (51) is simplified, respectively*

$$\begin{array}{ll} \max\limits_{u,\, x\in X} & \sum\limits_{j=1}^{K} p_j \, w_{jN}(x) \\ s.\,t. & u_{jk} \ge u_{j(k-1)} - r_{jk}^{(p)}(x), \\ & \gamma \ge u_{jk} \ge 0, \; u_{j0} = 0, \\ & j = \overline{1,K}, \; k = \overline{1,N}, \end{array} \tag{57}$$

$$\begin{array}{ll} \max\limits_{u,\, x\in X} & \sum\limits_{j=1}^{K} p_j \, w_{jN}(x) \\ s.\,t. & \frac{1}{N} \sum\limits_{k=1}^{N} \sum\limits_{j=1}^{K} p_j u_{jk} \le \gamma, \\ & u_{jk} \ge u_{j(k-1)} - r_{jk}^{(p)}(x), \\ & u_{jk} \ge 0, \; u_{j0} = 0, \\ & j = \overline{1,K}, \; k = \overline{1,N}. \end{array} \tag{58}$$

6.2 Efficient frontier

Efficient frontier is a central concept in Risk Management methodology. Suppose for every value of γ and risk profile χ , $x^*_\chi(\gamma)$ is an optimal solution to (51). In this case, efficient frontier is a curve expressing dependence of optimal portfolio expected reward $\sum\limits_{j=1}^{K} p_j\, w_{jN}(x^*_\chi(\gamma))$ on portfolio risk γ.

Proposition 11. *Efficient frontier* $\left(\gamma, \sum\limits_{j=1}^{K} p_j\, w_{jN}(x^*_\chi(\gamma))\right)$ *is a concave curve.*

Proof. Denoting $g(x) = \sum\limits_{j=1}^{K} p_j\, w_{jN}(x)$, we should establish that for any $\gamma_{1,2} \in [0,1]$ and $\tau \in [0,1]$

$$g(x^*_\chi(\tau\,\gamma_1 + (1-\tau)\gamma_2)) \geq \tau\, g(x^*_\chi(\gamma_1)) + (1-\tau)\, g(x^*_\chi(\gamma_2)).$$

According to the proof of Theorem 3, we have

$$\begin{array}{rcl} g(x^*_\chi(\gamma)) = & \max\limits_{x\in X,\, y} & g(x) \\ & s.\,t. & H(x,y) \leq \gamma, \end{array}$$

and using notation $G_\lambda(x,y) = g(x) - \lambda\, H(x,y)$, we obtain

$$g(x^*_\chi(\gamma)) = \min_{\lambda\geq 0}\ \max_{x\in X,\, y}\,(G_\lambda(x,y) + \lambda\gamma) = \min_{\lambda\geq 0}\ (G_\lambda(x(\lambda), y(\lambda)) + \lambda\gamma)\,.$$

Since expression $G_\lambda(x(\lambda),y(\lambda))+\lambda\gamma$ is linear with respect to γ, $\min\limits_{\lambda\geq 0}\ (G_\lambda(x(\lambda),y(\lambda)) + \lambda\gamma)$ is a concave function of γ. Indeed, $\min\limits_{\lambda\geq 0}\ (G_\lambda(x(\lambda),y(\lambda)) + \lambda(\tau\,\gamma_1 + (1-\tau)\gamma_2)) = \min\limits_{\lambda\geq 0}\ (\tau\,(G_\lambda(x(\lambda),y(\lambda)) + \lambda\gamma_1) + (1-\tau)\,(G_\lambda(x(\lambda),y(\lambda)) + \lambda\gamma_2)) \geq \tau\ \min\limits_{\lambda\geq 0}\ (G_\lambda(x(\lambda),y(\lambda)) + \lambda\gamma_1) + (1-\tau)\ \min\limits_{\lambda\geq 0}\ (G_\lambda(x(\lambda),y(\lambda)) + \lambda\gamma_2)$. This fact proves the proposition. □

An important characteristic for choosing an optimal portfolio on an efficient frontier is the ratio of the portfolio reward to the portfolio risk, so-called *risk-adjusted return*,

$$\rho_\chi(\gamma) = \gamma^{-1}\sum_{j=1}^{K} p_j\, w_{jN}(x^*_\chi(\gamma)). \tag{59}$$

A fund manager is interested in such a value of $\gamma \in [0,1]$, for which the risk-adjusted return $\rho_\chi(\gamma)$ is maximal. It is interpreted to be *the best balance between the risk accepted and the rate of return achieved.* According to Proposition 11, $\sum\limits_{j=1}^{K} p_j\, w_{jN}(x^*_\chi(\gamma))$ is concave, hence, ratio $\rho_\chi(\gamma)$ has a finite global maximum. Although $\rho_\chi(\gamma)$ is a nonlinear function with respect to γ, a problem for finding $\rho_\chi(\gamma)$ maximum and corresponding optimal γ is reduced to an LP.

Proposition 12. *The optimization problem* $\max\limits_{\gamma\in[0,1]} \rho_\chi(\gamma)$ *is reduced to LP*

$$\begin{array}{rl} \max\limits_{\tilde{u},\, v,\, \tilde{x}\in\tilde{X},\, \tilde{y},\, \tilde{z}} & \sum\limits_{j=1}^{K} p_j\ w_{iN}(\tilde{x}) \\ s.\,t. & \sum\limits_{i=1}^{L} \chi_i \left(\tilde{y}_i + \frac{1}{(1-\alpha_i)N}\sum\limits_{k=1}^{N}\sum\limits_{j=1}^{K} p_j\,\tilde{z}_{ijk}\right) \leq 1, \\ & \tilde{z}_{ijk} \geq \tilde{u}_{jk} - \tilde{y}_i, \\ & \tilde{u}_{jk} \geq \tilde{u}_{j(k-1)} - r^{(p)}_{jk}(\tilde{x}), \\ & \tilde{u}_{jk} \geq 0,\ \tilde{u}_{j0} = 0,\ \tilde{z}_{ijk} \geq 0, \\ & i = \overline{1,L},\ j = \overline{1,K},\ k = \overline{1,N}. \end{array} \tag{60}$$

If $\tilde{x}^*$ is an optimal solution for (60) then $\rho_\chi(\gamma^*) = \max\limits_{\gamma\in[0,1]} \rho_\chi(\gamma) = \sum\limits_{j=1}^{K} p_j\, w_{jN}(\tilde{x}^*)$, with optimal value $\gamma^* = 1 \Big/ \sum\limits_{l=0}^{m} \tilde{x}_l^*$ and corresponding optimal portfolio $x_l^* = \tilde{x}_l^*\,\gamma^*$, $l = \overline{0,m}$.

Proof. Since $\max\limits_{\gamma\in[0,1]} \rho_\chi(\gamma) = \max\limits_{\gamma\in[0,1]} \gamma^{-1} \sum\limits_{j=1}^{K} p_j\, w_{jN}(x_\chi^*(\gamma)) = \max\limits_{\gamma\in[0,1]} \max\limits_{x\in X_\chi} \gamma^{-1} \sum\limits_{j=1}^{K} p_j\, w_{jN}(x)$, where X_χ is the set of constraints in problem (51), the problem of $\max\limits_{\gamma\in[0,1]} \max\limits_{x\in X_\chi} \gamma^{-1} \sum\limits_{j=1}^{K} p_j\, w_{jN}(x)$ is reduced to LP (60) by changing variables $\tilde{x}_l = x_l/\gamma$, $\tilde{y}_i = y_i/\gamma$, $\tilde{u}_{kj} = u_{kj}/\gamma$, $\tilde{z}_{ijk} = z_{ijk}/\gamma$, $l = \overline{0,m}$, $i = \overline{1,L}$, $j = \overline{1,K}$, $k = \overline{1,N}$. Set $\tilde{X}$ may include additional variable $v = 1/\gamma$. For instance, a box constraint $x_{\min} \le x_l \le x_{\max}$ from the set X is transformed to $x_{\min}\, v \le \tilde{x}_l \le x_{\max}\, v$, which is contained in $\tilde{X}$. □

7 Drawdown measure in real-life portfolio optimization

7.1 Static asset allocation.

This section formulates and solves a real-life portfolio optimization problem with a constant (static) set of weights using drawdown measure.

A problem of dynamic weight allocation when asset (or a set of assets) is log-Brownian under a constraint on the worst equity drawdown was considered in several papers. First, a 1-dimensional case was solved by Grossman and Zhou as a mathematical programming problem in the outstanding article [12]. Then, the problem was generalized into a multi-dimensional case by Cvitanic and Karatzas in [7].

In difference to these works, we are looking to find a constant set of weights that optimizes a certain portfolio of assets, which are not assumed to have a log-brownian dynamics. This problem is stimulated by several important practical financial applications, particularly related to the so-called hedge-fund business.

A CTA company is a hedge fund that normally trades several (sometimes, more than a 100) futures markets simultaneously using some mathematical strategies that it believes have certain edge. These kinds of managers manage substantial assets as a part of all hedge funds, by some estimates, close to \$100 BN. Most of the CTA community trades the, so-called, long-term trend-following systems, but there are now multiple examples of short-term mean-reverting trading systems as well. These systems may be viewed as some functions of the individual futures market price realized prior to the present time. These strategies normally have a substantial smoothing-out effect on the futures prices and have close to stationary properties. Every CTA, then, has to allocate a certain portion of overall risk (or overall capital that it manages) to each and every "market". Due to a substantial level of stationarity of the strategies, each CTA calculates the weights according to a certain internal proprietary weight allocation procedure. Normally, this set remains fixed and does not change unless a certain market gets added or removed from the set, which normally happens when a new system is introduced, when a certain market disappears (like Deutsche Mark or French Franc in 1999), or a new market is being added. A standard practice in the CTA community is to use some version of the classical Markowitz mean-variance approach.

Another important example of static asset allocation comes form the so-called, Fund of Fund (FoF) business. In the recent several years this sector of hedge funds has experienced a substantial growth. A typical FoF manager gives allocations of its clients' capital to a set of pre-selected managers, normally between 5 and 25. It does so fairly infrequently, due to liquidity constraints imposed by managers themselves, but this is not the only reason. FoF views equity return streams as fairly stationary time series with some attractive return, risk, and correlation properties, which need some time to present themselves. Unless some unexpected event happens, the allocations are given for a substantial period of time, on average of 2 years or more. A group of analysts in a typical FoF is responsible for finding a constant set of weights, which will allow for a total portfolio of this FoF to be attractive to its clients.

Both of these typical cases are faced with a problem of finding a constant set of weights, which optimize their portfolios in a certain sense. The practical goal of this paper is to facilitate this process with a clear and statistically sound algorithm, which utilizes a newly designed set of risk-measures based on a notion of an equity drawdown.

Despite their known potential drawbacks, it is a well-accepted and, moreover, recommended practice [25], is to study historical back-tested strategy results of a hedge fund and, based on these results, obtain an estimate of the inherent risk using some risk measures. The only popular quantitative risk measure is VaR [25]. Various insufficiencies of the VaR measure are also widely known. We believe that the results developed in this work would allow a better understanding of how this can be achieved.

7.2 Historical data and scenario generation.

Even though scientists and engineers used certain simple versions of resampling procedures since 1930s, it was namely B. Efron (1979) who unified the disconnected ideas and resampling emerged as a robust method of estimating confidence intervals of some measurable functions over a statistical sample of data [9]. Method is particularly useful for the time series where getting other realizations of the data may be difficult or even impossible.

Bootstrap is a form of resampling the original data set bootstrap, which "resamples with replacement." Sometimes, the simplest version of it is called "nonparametric bootstrap."

The method originally was applied to some sociological and biological applications, staying in the shade for statistical, engineering and financial applications up until the 1990s. Due to their intrinsic "one realization only"-nature, the financial time series could be one of the best applications for resampling methods.

Within the financial applications, a strong particular interest in getting estimates of certain measurable quantities (such as rate of return, or standard deviation), comes from the development of trading systems. It is well known, that a problem of actual using over-fitted trading systems can possibly lead to substantial financial losses. Therefore, it is hard to underestimate the importance of a problem of discovering how overfit a particular trading system is. Among a few examples, one can mention a single asset trading system (for example, a system which trades a back-adjusted continuous 10-year U.S. Government Note futures contract), or, a more general portfolio optimization problem (allocation of weights between several assets in a portfolio subject to certain constraints).

In this article, we have considered a particular example of portfolio allocation. This example could be very relevant for such managers as global CTAs. Such managers are applying certain trading systems (very frequently, long-term trend-following systems) across a wide set of global futures markets attempting to take advantage of certain price movements occurring in these markets. Normally, after they are content with their trading system, they have to make a decision of allocating their portfolio risk between various markets.

In this example, we are given a set of sample paths of certain futures trading systems (in this particular case, some long-term trend-following system) as applied to a set of 32 different global futures markets. The system can be long, short or flat every market, always trades the same number of contracts with the average trade length between one to two months.

Here is a list of the markets that were traded (and their corresponding exchanges) with the system. Ticker symbols of FutureSource are used for their abbreviation. In alphabetical order of their ticker symbol:

1. AAO - The Australian All Ordinaries Index (OTC);
2. AD - Australian Dollar Currency Futures (CME);
3. AXB - Australian 10-Year Bond Futures (SFE);
4. BD - U.S. Long (30-Year) Treasury Bond Futures (CBT);
5. BP - British Pound Sterling Currency Futures (CME);
6. CD - Canadian Dollar Currency Futures (CME);
7. CP - Copper Futures (COMEX);
8. DGB - German 10-Year Bond (Bund) Futures (LIFFE);
9. DX - U.S. Dollar Index Currency Futures (FNX);
10. ED - 90-Day Euro Dollar Futures (CME);
11. EU - Euro Currency Futures (CME);
12. FV - U.S. 5-Year Treasury Note Futures (CBT);
13. FXADJY - Australian Dollar vs. Japanese Yen Cross Currency Forward (OTC);
14. FXBPJY - British Pound Sterling vs. Japanese Yen Cross Currency Forward (OTC);
15. FXEUBP - Euro vs. British Pound Sterling Cross Currency Forward (OTC);
16. FXEUJY - Euro vs. Japanese Yen Cross Currency Forward (OTC);
17. FXEUSF - Euro vs. Swiss Franc Cross Currency Forward (OTC);
18. FXNZUS - New Zealand Dollar Currency Forward (OTC);
19. FXUSSG - Singaporean Dollar Currency Forward (OTC);
20. FXUSSK - Swedish Krona Currency Forward (OTC);
21. GC - Gold 100 Oz. Futures (COMEX);
22. JY - Japanese Yen Currency Futures (CME);
23. LBT - Italian 10-Year Bond Forward (OTC);
24. LFT - FTSE-100 Index Futures (LIFFE);
25. LGL -Long Gilt (U.K. 10-Year Bond) Futures (LIFFE);
26. LML - Aluminum Futures (COMEX);
27. MNN - French Notional Bond Futures ();
28. SF - Swiss Franc Currency Futures (CME);
29. SI - Silver Futures (COMEX);
30. SJB - JGB (Japanese 10-Year Government Bond) Futures (TSE);
31. SNI - NIKKEI-225 Index Futures (SIMEX);
32. TY - 10-Year U.S. Government Bond Futures (CBT).

These markets cover most major asset classes traded through futures: fixed-income (short-term and long-term, both domestic and international), international equity indices, currencies and cross-currencies, and metals.

Thus, we were given a set of 32 time series with daily rates of return covering a period of time between 6/12/1995 and 12/13/1999. Time is measured in trading days only, with a convention of 5-day workdays per week, with adding previous day closing data for holidays with missing data.

A basic version of non-parametric bootstrap resampling consists in a process of creation of "children" samples from the original "father" sample in the following way. A process consists of filling out a "child" with father's daily rates of return in random order "with repacement", i.e. when one can pull out the same daily rate of return twice or more.

The noticed difficulty with such a way of constructing a resampled probability distribution function is that if the original "father" time-series contained certain auto-correlation structure, it will be totally lost in the "children"-resamples because of totally random mixing while generating the resamples. At the same time, namely those auto-correlation properties of the time series, if present, should be responsible for the trend-following systems having positive rate of return.

To remedy the situation, we will use a modification of a simple bootstrap resampling, which is called block-bootstrap resampling. Here is a brief description of the procedure.

First, we need to empirically study the correlation properties of the time-series involved. For all the data series, we have numerically calculated their auto-correlation coefficients $C(t)$ for the separation times t between 0 and 200 days. The cut-off of 200 trading days was chosen in such a way that the measurements of correlation coefficient would still have some statistical accuracy on a sample length of 1076 days used.

Next, we have empirically found a threshold for the absolute value of the auto-correlation coefficients (equal to 2.5%), above which the values of coefficient larger than this threshold are statistically significant. Then, moving in separation time t from 200 to 0, for all the time-series, we have found the first separation time, when $C(t) > 2.5\%$ first occurs. We have found that the maximal separation time, which will contain all the statistically significant correlation lengths of all the time series involved, is 100 trading days.

Next, instead of randomly picking an individual daily return from the original data series, we, now will be picking un-interchanged blocks of daily returns of length 100 trading days, starting from a random starting point. To ensure consistency across all the time series (or, otherwise, to preserve the cross-market correlation structure), we will be choosing the same starting point for all 32 time-series. That is, we will use the same random starting point for all the markets, then draw another starting point, and use it across all the markets again, etc. until we will fill-in the necessary number of "children"-resamples.

7.3 Numerical results

In this asset allocation problem, we impose additional ("technological") box constraints on portfolio weights $0.2 \leq x_i \leq 0.8$, $i = \overline{1, 32}$. This choice was dictated by the need to have the resultant margin-to-equity ratio in the account within admissible bounds, which are specific for a particular portfolio. These constraints, in this futures trading setup is analogous to the "fully-invested" condition from classical Sharpe-Markowitz theory [14], and it is namely this condition, which makes the efficient frontier strictly concave. In the absence of these constraints, the efficient frontier would be a straight line passing through (0,0), due to the virtually infinite leverage of these types of strategies. If all positions are equal to the lower bound 0.2, then the sum of the positions is $0.2 \times 32 = 6.4$ and the minimal leverage is 6.4. However, if all positions are equal to the upper bound 0.8, then the sum of the positions is $0.8 \times 32 = 25.6$ and the maximal leverage becomes 25.6. The optimal allocation of weights picks both the optimal leverage and proportions between instruments. Another subtle issue has to do with the stability of the optimal portfolios if the constraints are "too lax". It is a matter of empirical evidence that the more lax the constraints are – the better portfolio equity curve you can get through optimal mixing – and the less stable with respect to walk-forward analysis these results would be. The above set of constraints was empirically found to be both leading to sufficiently stable portfolios and allowing enough mixing of the individual equity curves.

We solved optimization problems (57), (58) and (51) with MaxDD , AvDD and 80%-CDD ($\alpha = 0.8$) measures, respectively, in the cases of 1 (historical), 100 and 300 sample paths generated for all 32 instruments. All optimization problems were solved using CPLEX package.

The graphs of efficient frontiers and tables with optimal portfolio configurations for optimization problems with MaxDD, AvDD, and 80%-CDD in all three cases: 1, 100 and 300 sample paths are presented by Figures 6, 8, 10 and Tables 1–9, respectively. We, also, enclose the risk-adjusted returns (annualized rate of return divided by the corresponding value of a risk measure) for each of these cases, see Figures 7, 9 and 11. The solutions achieving maximal risk-adjusted returns are boldfaced, see Tables 1–9.

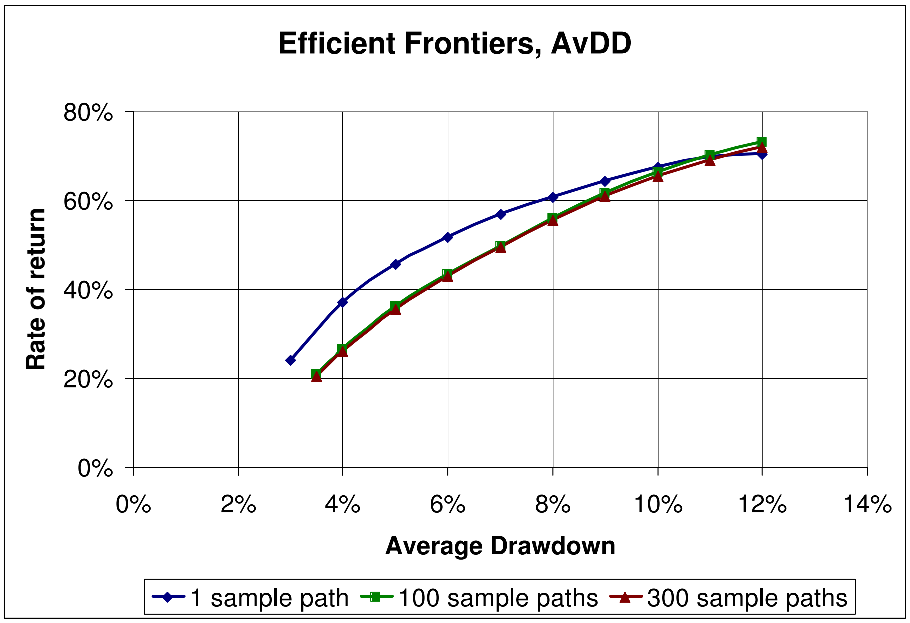

Figure 6: Efficient frontiers for Average Drawdown problem.

8 Conclusions

We introduced a new CDD measure, which, we believe, is useful for the practical portfolio management. This measure is similar to CVaR risk measure and has the MaxDD and AvDD measures as its limiting cases. The CDD possesses all properties of a deviation measure and may be considered as one of its generalizations to a dynamic case. We developed the optimization techniques that efficiently solve the portfolio allocation problem with CDD, MaxDD and AvDD measures. We have posed and for a real-life example, solved a portfolio allocation problem. These developments, if implemented in a managed accounts' environment will allow a trading or risk manager to allocate risk according to his/her personal assessment of extreme drawdowns and their duration on his/her portfolio equity.

We believe that however attractive the MaxDD measure is, the solutions produced with using this measure in portfolio optimization may have a significant statistical error because the decision is based on a single observation of the maximal loss. Having a CDD family of risk measures allows a risk manager to control the worst $(1-\alpha) * 100\%$ of drawdowns, and due to statistical averaging within that range, to get a better predictive power of this risk measure in the future, leading to a more stable portfolio. Our studies indicate that when considering CDD with an appropriate level (e.g., $\alpha = 0.8$, i.e., optimizing over the 20% of the worst drawdowns), one can get a more stable weights allocation

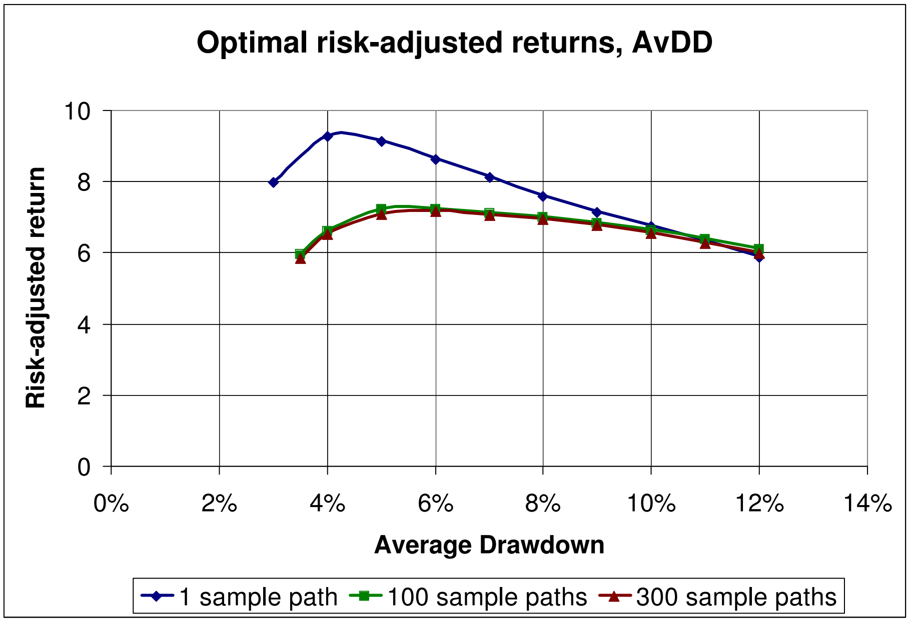

Figure 7: Optimal risk-adjusted returns for Average Drawdown problem.

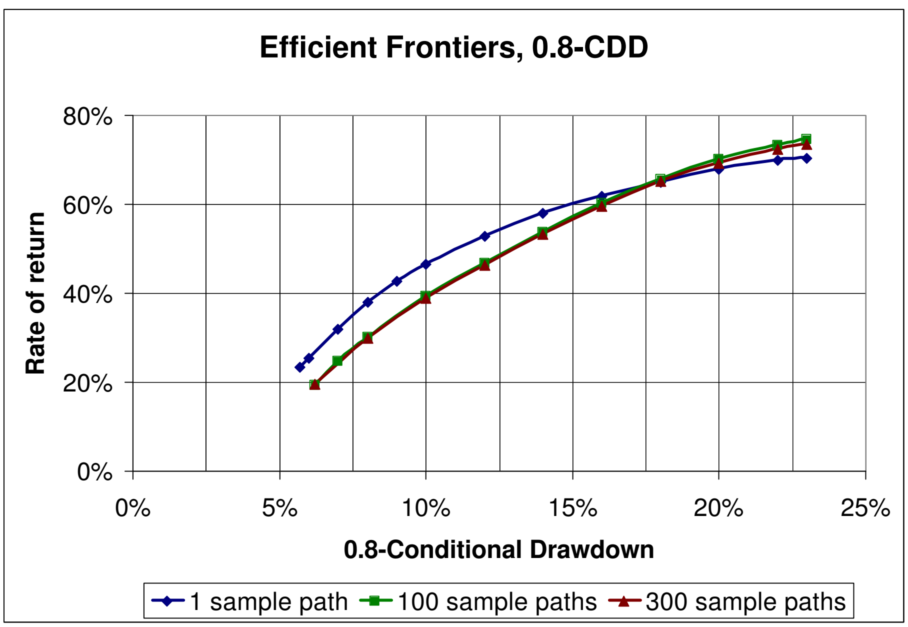

Figure 8: Efficient frontiers for 0.8-Conditional Drawdown problem.

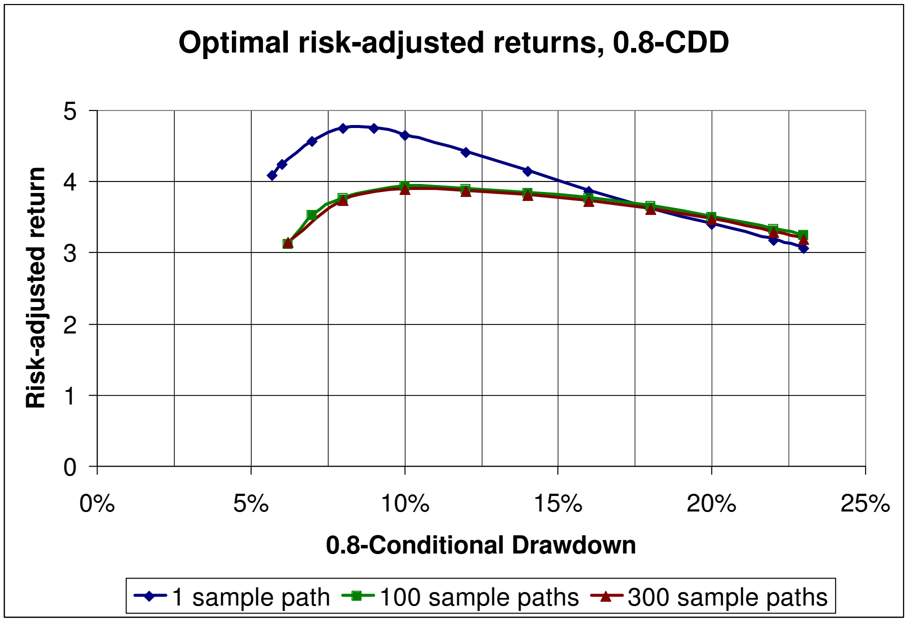

Figure 9: Optimal risk-adjusted returns for 0.8-Conditional Drawdown problem.

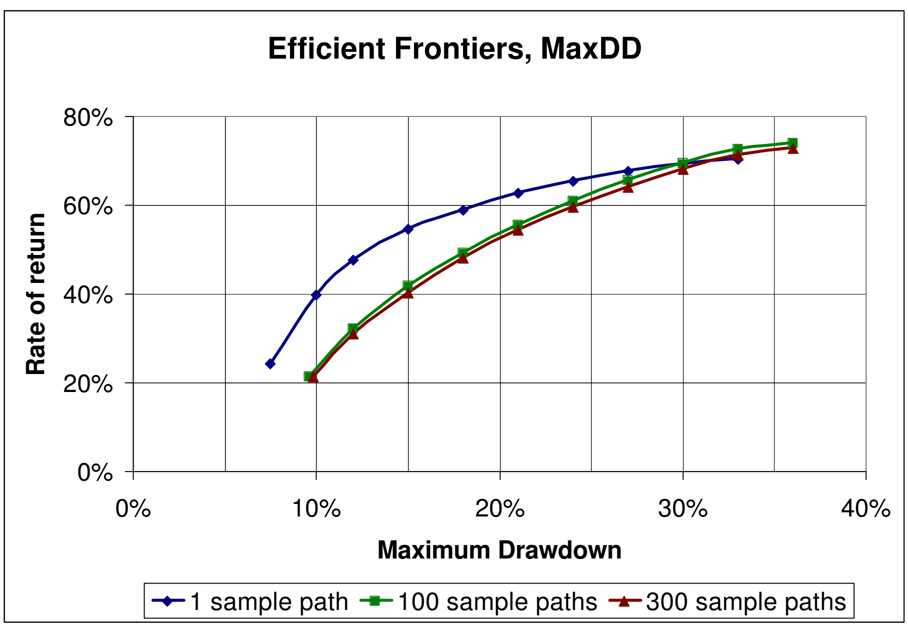

Figure 10: Efficient frontiers for Maximum Drawdown problem.

Rate of return, %	*23.9*	***37.1***	*45.7*	*51.7*	*56.8*	*60.7*	*64.3*	*67.4*	*69.6*	*70.4*
AvDD, %	*3.0*	***4.0***	*5.0*	*6.0*	*7.0*	*8.0*	*9.0*	*10.0*	*11.0*	*12.0*
Risk-adj. return	*7.98*	***9.27***	*9.13*	*8.62*	*8.12*	*7.59*	*7.15*	*6.74*	*6.33*	*5.86*
AD	0.20	**0.20**	0.20	0.20	0.20	0.63	0.46	0.80	0.80	0.80
BD	0.20	**0.20**	0.39	0.20	0.20	0.20	0.80	0.80	0.80	0.80
BP	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.65
CD	0.20	**0.64**	0.74	0.80	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.20	**0.20**	0.20	0.20	0.20	0.62	0.71	0.80	0.80	0.80
DX	0.20	**0.20**	0.20	0.80	0.80	0.80	0.80	0.80	0.80	0.80
ED	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	**0.78**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	**0.20**	0.20	0.29	0.52	0.76	0.80	0.80	0.80	0.80
FXBPJY	0.20	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.49	**0.51**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.79	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.27	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	**0.47**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.70	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.79	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80	0.80	0.80
GC	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.80	0.80
JY	0.20	**0.41**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LIFT	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	**0.20**	0.67	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LIIB	0.20	**0.20**	0.20	0.22	0.73	0.80	0.80	0.80	0.80	0.80
LMAL	0.20	**0.20**	0.20	0.45	0.80	0.80	0.80	0.80	0.80	0.80
MANB	0.20	**0.20**	0.20	0.20	0.20	0.20	0.77	0.80	0.80	0.80
SF	0.20	**0.20**	0.20	0.20	0.20	0.39	0.73	0.80	0.80	0.80
SFAO	0.20	**0.20**	0.43	0.76	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.22	**0.53**	0.44	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.24	0.80
SIJB	0.20	**0.20**	0.20	0.32	0.46	0.47	0.50	0.77	0.80	0.80
SINI	0.20	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.25	0.80	0.80
UXBU	0.20	**0.54**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 1: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Average Drawdown problem with 1 sample path.

Rate of return, %	*20.8*	*26.4*	*36.1*	***43.4***	*49.7*	*55.8*	*61.5*	*66.3*	*70.1*	*73.1*
AvDD, %	*3.5*	*4.0*	*5.0*	***6.0***	*7.0*	*8.0*	*9.0*	*10.0*	*11.0*	*12.0*
Risk-adj. return	*5.94*	*6.61*	*7.22*	***7.23***	*7.11*	*6.98*	*6.83*	*6.63*	*6.38*	*6.09*
AD	0.20	0.20	0.20	**0.22**	0.30	0.36	0.77	0.80	0.80	0.80
BD	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.25	0.80	0.80
BP	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
CD	0.44	0.38	0.77	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.20	0.20	0.36	**0.50**	0.53	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.48
ED	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.35	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.20	0.20	**0.26**	0.46	0.54	0.64	0.70	0.79	0.80
FXBPJY	0.20	0.29	0.69	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	0.22	**0.45**	0.75	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.48	0.45	0.20	**0.20**	0.21	0.51	0.80	0.80	0.80	0.80
FXEUSF	0.24	0.47	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	0.20	**0.20**	0.44	0.74	0.80	0.80	0.80	0.80
FXUSSG	0.26	0.36	0.74	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.56	0.73	0.72	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.38	0.80	0.80	0.80
GC	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.34	0.78	0.80
JY	0.20	0.20	0.20	**0.20**	0.36	0.62	0.80	0.80	0.80	0.80
LIFT	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	0.20	**0.20**	0.24	0.51	0.68	0.80	0.80	0.80
LIIB	0.20	0.20	0.20	**0.20**	0.28	0.44	0.72	0.80	0.80	0.80
LMAL	0.20	0.20	0.20	**0.20**	0.20	0.20	0.23	0.38	0.42	0.80
MANB	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.80	0.80	0.80
SF	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.64	0.80
SFAO	0.20	0.31	0.42	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	0.20	**0.45**	0.58	0.80	0.80	0.80	0.80	0.80
SI	0.21	0.46	0.70	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.20	0.20	**0.24**	0.37	0.32	0.46	0.64	0.80	0.80
SINI	0.20	0.20	0.20	**0.51**	0.77	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.60
UXBU	0.21	0.58	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80

Table 2: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Average Drawdown problem with 100 sample paths.

Rate of return, %	*20.5*	*26.1*	*35.4*	***42.9***	*49.4*	*55.4*	*60.8*	*65.3*	*69.0*	*71.9*
AvDD, %	*3.5*	*4.0*	*5.0*	***6.0***	*7.0*	*8.0*	*9.0*	*10.0*	*11.0*	*12.0*
Risk-adj. return	*5.85*	*6.52*	*7.08*	***7.16***	*7.06*	*6.93*	*6.76*	*6.53*	*6.27*	*5.99*
AD	0.20	0.20	0.20	**0.29**	0.26	0.51	0.80	0.80	0.80	0.80
BD	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.80	0.80
BP	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
CD	0.24	0.32	0.72	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.20	0.20	0.47	**0.50**	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.23	0.20	0.63
ED	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.37	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.20	0.20	**0.20**	0.36	0.42	0.49	0.60	0.69	0.80
FXBPJY	0.20	0.20	0.66	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	0.27	**0.55**	0.77	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.62	0.73	0.29	**0.33**	0.37	0.69	0.80	0.80	0.80	0.80
FXEUSF	0.23	0.52	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	0.20	**0.20**	0.46	0.62	0.80	0.80	0.80	0.80
FXUSSG	0.22	0.33	0.74	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.59	0.69	0.65	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.24	0.80	0.80	0.80
GC	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.22	0.62	0.72
JY	0.20	0.20	0.20	**0.20**	0.45	0.70	0.80	0.80	0.80	0.80
LIFT	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	0.20	**0.20**	0.27	0.56	0.57	0.80	0.80	0.80
LIIB	0.20	0.20	0.20	**0.20**	0.20	0.45	0.72	0.80	0.80	0.80
LMAL	0.20	0.20	0.20	**0.22**	0.22	0.20	0.20	0.41	0.46	0.80
MANB	0.20	0.20	0.20	**0.20**	0.20	0.20	0.70	0.80	0.80	0.80
SF	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.35	0.80	0.80
SFAO	0.20	0.25	0.38	**0.63**	0.78	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	0.20	**0.40**	0.63	0.80	0.80	0.80	0.80	0.80
SI	0.21	0.37	0.52	**0.72**	0.80	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.20	0.20	**0.27**	0.32	0.38	0.50	0.69	0.80	0.80
SINI	0.20	0.20	0.22	**0.67**	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.50
UXBU	0.29	0.69	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80

Table 3: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Average Drawdown problem with 300 sample paths.

Rate of return, %	*23.3*	*25.5*	*31.9*	***38.0***	*46.6*	*52.9*	*57.9*	*61.8*	*65.0*	*70.3*
0.8-CDD, %	*5.7*	*6.0*	*7.0*	***8.0***	*10.0*	*12.0*	*14.0*	*16.0*	*18.0*	*23.0*
Risk-adj. return	*4.09*	*4.24*	*4.56*	***4.75***	*4.66*	*4.41*	*4.14*	*3.86*	*3.61*	*3.06*
AD	0.20	0.20	0.20	**0.20**	0.20	0.20	0.42	0.55	0.33	0.80
BD	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.44	0.80	0.80
BP	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.56
CD	0.20	0.20	0.20	**0.20**	0.27	0.80	0.80	0.80	0.80	0.80
CP	0.20	0.20	0.20	**0.20**	0.20	0.20	0.80	0.80	0.80	0.80
DX	0.20	0.20	0.20	**0.20**	0.20	0.72	0.80	0.80	0.80	0.80
ED	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.20	0.20	**0.50**	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.25	0.26	**0.25**	0.30	0.37	0.29	0.36	0.80	0.80
FXBPJY	0.77	0.80	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.33	0.39	**0.53**	0.76	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.80	0.80	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.20	0.20	**0.52**	0.27	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.25	0.40	**0.68**	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.42	0.47	0.65	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.20	0.20	0.33	**0.43**	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80
GC	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80
JY	0.20	0.25	0.60	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80
LIFT	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	0.20	**0.20**	0.20	0.20	0.80	0.80	0.80	0.80
LIIB	0.20	0.20	0.49	**0.44**	0.74	0.80	0.80	0.80	0.80	0.80
LMAL	0.20	0.20	0.20	**0.20**	0.20	0.20	0.26	0.59	0.48	0.80
MANB	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.80	0.80	0.80
SF	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.61	0.80
SFAO	0.20	0.20	0.20	**0.20**	0.61	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	0.20	**0.20**	0.63	0.80	0.80	0.80	0.80	0.80
SI	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80
SIJB	0.20	0.21	0.20	**0.29**	0.29	0.38	0.60	0.80	0.80	0.80
SINI	0.47	0.58	0.60	**0.67**	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80
UXBU	0.20	0.23	0.52	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80

Table 4: Optimal rates of returns, risk-adjusted returns and portfolio configurations for 0.8-Conditional Drawdown problem with 1 sample path.

Rate of return, %	*19.4*	*24.7*	*30.1*	***39.3***	*46.7*	*53.6*	*60.1*	*65.7*	*70.0*	*74.5*
0.8-CDD, %	*6.2*	*7.0*	*8.0*	***10.0***	*12.0*	*14.0*	*16.0*	*18.0*	*20.0*	*23.0*
Risk-adj. return	*3.12*	*3.53*	*3.76*	***3.93***	*3.89*	*3.83*	*3.76*	*3.65*	*3.50*	*3.24*
AD	0.20	0.20	0.20	**0.20**	0.21	0.29	0.62	0.80	0.80	0.80
BD	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.74	0.80
BP	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.55
CD	0.20	0.20	0.20	**0.33**	0.47	0.58	0.75	0.80	0.80	0.80
CP	0.20	0.20	0.20	**0.41**	0.46	0.62	0.80	0.80	0.80	0.80
DX	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.28	0.80
ED	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.24	0.50	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.20	0.20	**0.20**	0.23	0.36	0.45	0.60	0.79	0.80
FXBPJY	0.20	0.40	0.51	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	0.20	**0.43**	0.62	0.79	0.80	0.80	0.80	0.80
FXEUJY	0.39	0.54	0.49	**0.47**	0.77	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.43	0.67	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	0.20	**0.20**	0.40	0.75	0.80	0.80	0.80	0.80
FXUSSG	0.20	0.39	0.52	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.51	0.74	0.73	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.24	0.67	0.80	0.80
GC	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.24	0.80
JY	0.20	0.20	0.20	**0.20**	0.25	0.41	0.70	0.80	0.80	0.80
LIFT	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	0.20	**0.20**	0.20	0.35	0.72	0.80	0.80	0.80
LIIB	0.20	0.20	0.20	**0.23**	0.43	0.57	0.75	0.80	0.80	0.80
LMAL	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.37	0.80
MANB	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.77	0.80	0.80
SF	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.24	0.80	0.80
SFAO	0.20	0.20	0.32	**0.53**	0.78	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	0.20	**0.20**	0.26	0.39	0.49	0.74	0.80	0.80
SI	0.20	0.20	0.26	**0.55**	0.72	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.20	0.20	**0.40**	0.48	0.58	0.69	0.80	0.80	0.80
SINI	0.20	0.20	0.20	**0.24**	0.58	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.80
UXBU	0.20	0.47	0.78	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80

Table 5: Optimal rates of returns, risk-adjusted returns and portfolio configurations for 0.8-Conditional Drawdown problem with 100 sample paths.

Rate of return, %	*19.5*	*29.8*	***38.9***	*46.3*	*53.3*	*59.6*	*65.1*	*69.3*	*72.4*	*73.5*
0.8-CDD, %	*6.2*	*8.0*	***10.0***	*12.0*	*14.0*	*16.0*	*18.0*	*20.0*	*22.0*	*23.0*
Risk-adj. return	*3.14*	*3.73*	***3.89***	*3.86*	*3.80*	*3.73*	*3.62*	*3.46*	*3.29*	*3.19*
AD	0.20	0.20	**0.20**	0.37	0.36	0.72	0.80	0.80	0.80	0.80
BD	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.72	0.80	0.80
BP	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.66
CD	0.20	0.20	**0.33**	0.37	0.47	0.71	0.80	0.80	0.80	0.80
CP	0.20	0.25	**0.48**	0.56	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	**0.20**	0.20	0.20	0.20	0.38	0.62	0.80	0.80
ED	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.57	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	0.20	**0.20**	0.27	0.40	0.49	0.57	0.72	0.80	0.80
FXBPJY	0.20	0.50	**0.77**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	**0.47**	0.64	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.58	0.66	**0.64**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.72	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	**0.20**	0.32	0.63	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.21	0.49	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.57	0.68	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.44	0.80	0.80	0.80
GC	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.62	0.80
JY	0.20	0.20	**0.20**	0.40	0.59	0.80	0.80	0.80	0.80	0.80
LIFT	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	**0.20**	0.20	0.39	0.72	0.80	0.80	0.80	0.80
LIIB	0.20	0.20	**0.20**	0.33	0.49	0.70	0.80	0.80	0.80	0.80
LMAL	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.39	0.80	0.80
MANB	0.20	0.20	**0.20**	0.20	0.20	0.20	0.80	0.80	0.80	0.80
SF	0.20	0.20	**0.20**	0.20	0.20	0.20	0.35	0.80	0.80	0.80
SFAO	0.20	0.27	**0.50**	0.70	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	**0.20**	0.42	0.51	0.68	0.80	0.80	0.80	0.80
SI	0.20	0.20	**0.36**	0.47	0.73	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.20	**0.41**	0.49	0.56	0.68	0.80	0.80	0.80	0.80
SINI	0.20	0.20	**0.33**	0.71	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.62	0.80
UXBU	0.20	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 6: Optimal rates of returns, risk-adjusted returns and portfolio configurations for 0.8-Conditional Drawdown problem with 300 sample paths.

Rate of return, %	*24.3*	***39.7***	*47.6*	*54.6*	*58.8*	*62.6*	*65.5*	*67.6*	*69.3*	*70.4*
MaxDD, %	*7.5*	***10.0***	*12.0*	*15.0*	*18.0*	*21.0*	*24.0*	*27.0*	*30.0*	*33.0*
Risk-adj. return	*3.24*	***3.97***	*3.97*	*3.64*	*3.27*	*2.98*	*2.73*	*2.50*	*2.31*	*2.13*
AD	0.20	**0.20**	0.20	0.20	0.20	0.20	0.49	0.80	0.80	0.80
BD	0.20	**0.20**	0.20	0.20	0.20	0.20	0.80	0.80	0.80	0.80
BP	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.73
CD	0.20	**0.20**	0.20	0.20	0.80	0.80	0.80	0.80	0.80	0.80
CP	0.20	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.33	**0.20**	0.55	0.80	0.80	0.80	0.80	0.80	0.80	0.80
ED	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	**0.20**	0.48	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXADJY	0.20	**0.57**	0.54	0.49	0.79	0.80	0.80	0.80	0.80	0.80
FXBPJY	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	**0.79**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.63	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.31	**0.66**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	**0.70**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.20	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.80	0.80
GC	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.72	0.80
JY	0.20	**0.41**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LIFT	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	**0.20**	0.20	0.20	0.20	0.80	0.80	0.80	0.80	0.80
LIIB	0.20	**0.20**	0.20	0.80	0.80	0.80	0.80	0.80	0.80	0.80
LMAL	0.20	**0.20**	0.20	0.23	0.20	0.20	0.20	0.20	0.20	0.80
MANB	0.20	**0.20**	0.20	0.20	0.20	0.33	0.80	0.80	0.80	0.80
SF	0.20	**0.20**	0.20	0.20	0.32	0.80	0.80	0.80	0.80	0.80
SFAO	0.48	**0.66**	0.77	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	**0.20**	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.80	0.80
SIJB	0.20	**0.20**	0.20	0.55	0.80	0.80	0.80	0.80	0.80	0.80
SINI	0.20	**0.37**	0.72	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.73	0.80	0.80
UXBU	0.20	**0.47**	0.68	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 7: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Maximum Drawdown problem with 1 sample path.

Rate of return, %	*21.3*	*32.2*	***41.7***	*49.2*	*55.5*	*60.8*	*65.6*	*69.5*	*72.6*	*74.0*
MaxDD, %	*9.6*	*12.0*	***15.0***	*18.0*	*21.0*	*24.0*	*27.0*	*30.0*	*33.0*	*36.0*
Risk-adj. return	*2.21*	*2.68*	***2.78***	*2.74*	*2.64*	*2.53*	*2.43*	*2.32*	*2.20*	*2.06*
AD	0.20	0.20	**0.46**	0.43	0.25	0.20	0.70	0.80	0.80	0.80
BD	0.20	0.20	**0.20**	0.20	0.20	0.38	0.80	0.80	0.80	0.80
BP	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.35	0.80
CD	0.20	0.20	**0.20**	0.20	0.45	0.49	0.73	0.74	0.80	0.80
CP	0.12	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.80	0.80	0.80
ED	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.44	**0.66**	0.58	0.80	0.80	0.50	0.80	0.80	0.80
FXADJY	0.20	0.30	**0.33**	0.42	0.71	0.80	0.80	0.80	0.80	0.80
FXBPJY	0.20	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.20	**0.61**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.20	0.20	**0.23**	0.20	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.44	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	**0.20**	0.51	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.34	0.49	**0.73**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.74	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.59	0.80	0.80
GC	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.21	0.39
JY	0.20	0.20	**0.46**	0.30	0.29	0.63	0.70	0.80	0.80	0.80
LIFT	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	**0.20**	0.69	0.80	0.80	0.80	0.80	0.80	0.80
LIIB	0.20	0.20	**0.20**	0.20	0.40	0.80	0.80	0.80	0.80	0.80
LMAL	0.20	0.20	**0.20**	0.29	0.20	0.20	0.20	0.20	0.20	0.64
MANB	0.20	0.20	**0.20**	0.32	0.20	0.29	0.28	0.20	0.80	0.80
SF	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.55	0.80	0.80
SFAO	0.35	0.36	**0.41**	0.74	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	**0.20**	0.38	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.60	0.74	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.20	**0.20**	0.20	0.20	0.28	0.80	0.80	0.80	0.80
SINI	0.20	0.20	**0.65**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	**0.20**	0.20	0.21	0.48	0.71	0.76	0.80	0.80
UXBU	0.20	0.66	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 8: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Maximum Drawdown problem with 100 sample paths.

Rate of return, %	*21.3*	*31.1*	***40.3***	*48.1*	*54.4*	*59.6*	*64.1*	*68.1*	*71.3*	*72.7*
MaxDD, %	*9.8*	*12.0*	***15.0***	*18.0*	*21.0*	*24.0*	*27.0*	*30.0*	*33.0*	*36.0*
Risk-adj. return	*2.18*	*2.59*	***2.69***	*2.67*	*2.59*	*2.49*	*2.38*	*2.27*	*2.16*	*2.02*
AD	0.20	0.20	**0.37**	0.63	0.36	0.20	0.46	0.80	0.80	0.80
BD	0.20	0.20	**0.20**	0.20	0.20	0.20	0.80	0.80	0.80	0.80
BP	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.79
CD	0.20	0.20	**0.20**	0.20	0.23	0.35	0.50	0.76	0.80	0.80
CP	0.50	0.76	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
DX	0.20	0.20	**0.20**	0.20	0.20	0.20	0.25	0.80	0.80	0.80
ED	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
EU	0.20	0.42	**0.49**	0.54	0.80	0.76	0.80	0.80	0.80	0.80
FXADJY	0.20	0.25	**0.40**	0.52	0.64	0.80	0.80	0.80	0.80	0.80
FXBPJY	0.29	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUBP	0.20	0.21	**0.51**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXEUJY	0.20	0.20	**0.24**	0.64	0.80	0.80	0.80	0.80	0.80	0.80
FXEUSF	0.20	0.21	**0.50**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXNZUS	0.20	0.20	**0.37**	0.28	0.68	0.80	0.80	0.80	0.80	0.80
FXUSSG	0.26	0.48	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FXUSSK	0.80	0.80	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
FY	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.54	0.72	0.80
GC	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
JY	0.20	0.20	**0.27**	0.26	0.21	0.61	0.80	0.80	0.80	0.80
LIFT	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.20	0.20	0.20
LIGI	0.20	0.20	**0.20**	0.27	0.80	0.80	0.80	0.80	0.80	0.80
LIIB	0.20	0.20	**0.20**	0.20	0.51	0.80	0.80	0.80	0.80	0.80
LMAL	0.20	0.20	**0.20**	0.37	0.20	0.20	0.20	0.20	0.26	0.70
MANB	0.20	0.20	**0.20**	0.55	0.23	0.31	0.31	0.20	0.80	0.80
SF	0.20	0.20	**0.20**	0.20	0.20	0.20	0.20	0.56	0.80	0.80
SFAO	0.27	0.37	**0.42**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SFBD	0.20	0.20	**0.36**	0.26	0.80	0.80	0.80	0.80	0.80	0.80
SI	0.20	0.59	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
SIJB	0.20	0.25	**0.34**	0.20	0.24	0.46	0.80	0.80	0.80	0.80
SINI	0.20	0.41	**0.58**	0.80	0.80	0.80	0.80	0.80	0.80	0.80
TY	0.20	0.20	**0.20**	0.20	0.20	0.48	0.67	0.70	0.80	0.80
UXBU	0.20	0.61	**0.80**	0.80	0.80	0.80	0.80	0.80	0.80	0.80

Table 9: Optimal rates of returns, risk-adjusted returns and portfolio configurations for Maximum Drawdown problem with 300 sample paths.

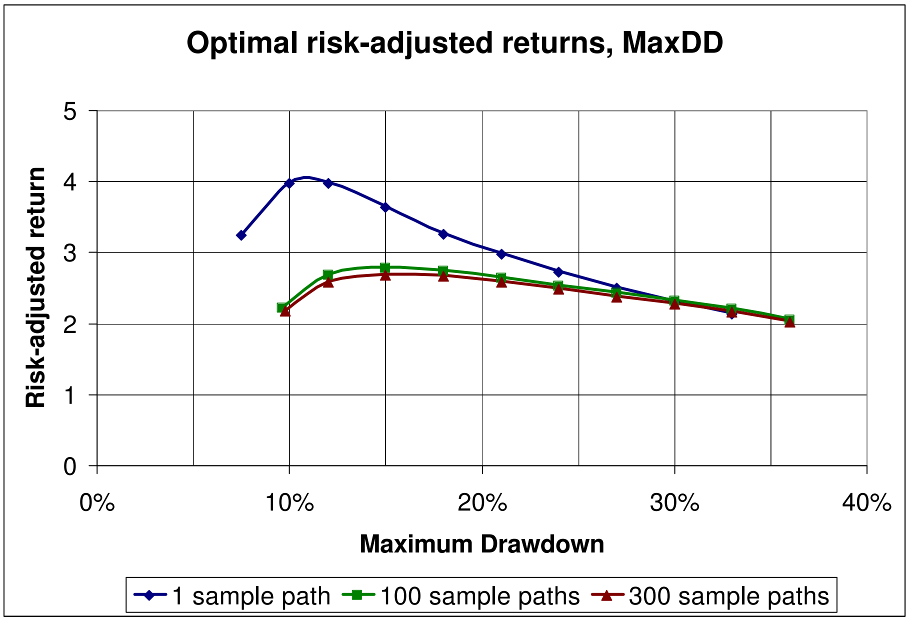

Figure 11: Optimal risk-adjusted returns for Maximum Drawdown problem.

than that produced with using MaxDD measure.

Numerical results of the considered real-life asset allocation problem with CDD measure draw the following conclusions

- The statistical accuracy is already sufficient for the case of 100 sample paths, i.e. difference between 100-sample path solutions and 300-sample paths solutions is negligible.
- For most of the allowable risk values (across all the risk measures considered), the efficient frontier for stochastic (re-sampled) solutions lies below and is less concave than the so-called historical, or 1-scenario, efficient frontier. Only at the riskiest end of the efficient frontier, the efficient frontiers either converge to one another or intersect. This means that only for the riskiest portfolios, the stochastic, or, re-sampled solutions, provide an improvement to the risk-adjusted returns.
- The risk adjusted returns, especially at the optimal (maximal risk-adjusted return) point on the efficient frontier are, however, uniformly smaller for the re-sampled, or, stochastic solutions. On average, re-sampled optimal risk-adjusted returns solutions turn out to be 20% to 30% worse than those predicted by 1-path historical solutions. This result lies in synch with the wide-spread notion that using only one (actually realized) history price path may lead (and probably does) to overstated and overfited results which may not realize on average in the future. Even though the results of the re-sampled, or, stochastic optimization may lead to worse optimal solutions, but those solutions will be more trustworthy.
- Analyzing the difference between the optimal historical and stochastic solutions (the vectors of instruments weights), one can find that the solutions are substantially different: for example, as vectors in their 32-dimensional space, they have very different lengths and are tilted with respect to one another at a large angle. The Euclidian norm of the stochastic optimal solution is, on average 50%, smaller than that for the historical optimal solution, and the estimate for the angle for our particular case is 50 degrees.

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References

[1] C. Acerbi, D. Tasche, On the Coherence of Expected Shortfall, *Journal of Banking and Finance* **26** (2002), 1487–1503.

[2] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent Measures of Risk, *Mathematical Finance* **9** (1999), 203–227.

[3] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, H. Ku, Multiperiod Risk and Coherent Multiperiod Risk Measurement, preprint 2002.

[4] F. Delbaen, Coherent Risk Measures on General Probability Spaces, preprint 2000.

[5] F. Delbaen, Draft: Coherent Risk Measures, lecture notes, Pisa, 2000.

[6] A. Chekhlov, S. Uryasev, M. Zabarankin, Portfolio Optimization with Drawdown Constraints, *Asset and Liability Management Tools*, (B. Scherer ed.), Risk Books, London, 2003, pp. 263–278.

[7] J. Cvitanic, I. Karatzas, On Portfolio Optimization Under "Drawdown" Constraints, *IMA Lecture Notes in Mathematics & Applications*, **65** (1995), 77–88.

[8] R.S. Dembo, A.J. King, Tracking Models and the Optimal Regret Distribution in Asset Allocation, *Applied Stochastic Models and Data Analysis*, Vol. **8** (1992), 151–157.

[9] Bradley Efron, Robert J. Tibshirani, *An Introduction to the Bootstrap*, Chapman & Hall/CRC, New York, 1993.

[10] P. Jorion, *Value at Risk : A New Benchmark for Measuring Derivatives Risk*, Irwin Professional Publisher, 1996.

[11] R.C. Grinold, R.N. Kahn, *Active Portfolio Management*, McGraw-Hill, New York, 1999.

[12] S.J. Grossman, Z. Zhou, Optimal Investment Strategies for Controlling Drawdowns, *Mathematical Finance*, **3** (3) (1993), 241–276.

[13] H. Konno, H. Yamazaki, Mean Absolute Deviation Portfolio Optimization Model and Its Application to Tokyo Stock Market, *Management Science*, **37** (1991), 519–531.

[14] H.M. Markowitz, Portfolio Selection, *Journal of Finance*, Vol. **7** (1952), 1, 77–91.

[15] H. Mausser, D. Rosen, Beyond VaR: From Measuring Risk to Managing Risk, *ALGO Research Quarterly*, Vol. **1** (1999), No. 2, 5–20.

[16] G.C. Pflug, Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk, in *Probabilistic Constrainted Optimization: Methodology and Applications* (S.P. Uryasev, ed.), Kluwer, 2000, 278–287.

[17] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[18] R.T. Rockafellar, S.P. Uryasev, Optimization of Conditional Value-at-Risk, *Journal of Risk* **2** (2000), 21–42.

[19] R.T. Rockafellar, S.P. Uryasev, Conditional Value-at-Risk for General Loss Distributions, *Journal of Banking and Finance*, **26** (2002), 1443–1471.

[20] R.T. Rockafellar, S. Uryasev, M. Zabarankin, Deviation Measures in Risk Analysis and Optimization, *Research Report 2002-7,* ISE Dept., University of Florida, 2002.

[21] R.T. Rockafellar, S. Uryasev, M. Zabarankin, Deviation Measures in Generalized Linear Regression, *Research Report 2002-9,* ISE Dept., University of Florida, 2002.

[22] R.T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[23] M.R. Young, A Minimax Portfolio Selection Rule with Linear Programming Solution. *Management Science,* Vol. **44** (1998), No. 5, 673–683.

[24] W.T. Ziemba, J.M. Mulvey, (Eds.) *Worldwide Asset and Liability Modeling*, Cambridge University Press, 1998.

[25] Sound practices for hedge fund managers, A response to the President's Working Group (PWG) report on LTCM crisis by the leading hedge funds: Caxton Corporation, Kingdon Capital LLC, Moore Capital Management Inc, Soros Fund Management LLC, Tudor Investment Corporation, February 2000.